

УПРАВЛІННЯ У ТЕХНІЧНИХ СИСТЕМАХ

УПРАВЛЕНИЕ В ТЕХНИЧЕСКИХ СИСТЕМАХ

CONTROL IN TECHNICAL SYSTEMS

UDC 331.101.1:629.735.083:656.7.086(045)

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OPTIMAL CONTROLLING PATH DETERMINATION WITH THE HELP OF HYBRID OPTIONAL FUNCTIONS DISTRIBUTIONS

Context. The problem of the determination of the optimal value of the augmentation coefficient of a proportional governor included into an inertness-less linear object control system on the basis of a synthesized model is solved. The object of the presented study is the optimal control process.

Objective. The goal of the work is a creation of a method for a problematic situation of the optimum definition, evaluation, and determination solving at the control system.

Method. A rough model of the phenomenon, and simplified dependence of optimal controlling trajectory upon the cost, of control in an inertness-less linear controlling system equipped with a proportional governor are proposed. The accuracy of the behavior of the investigated linear object of control has been chosen in the given consideration as an initial target value which needs to be minimized. The method of the model building with regards to an expenditures principle is offered. It provides taking into account the cost of controlling process. It allows finding the optimal controlling value on the multi-optional basis. There applied a certain analogue to the subjective entropy maximum principle of the subjective analysis in order to obtain a specific optimal distributions for the objective value in the view of the composed functional. The method of the uncertainty degree of the options extremization is improved by a continuous optional value introduction that allows forming the value distribution density. The optional synthesized model of the control process is built.

Results. The developed theoretical models allow obtaining, and have been implemented in, finding the hybrid optional density as an optimal solution of a variational problem with two independent variables, which maximal value is the sought optimal controlling path delivering minimum to the integrated expenses pertaining with the process.

Conclusions. The numerical experiments on the proposed methods studying in the problem of optimization are conducted. The discovered dependencies are substantiated as a result of these experiments. Their use in practice makes it possible, and is recommended, to carryout optimal control in the described systems. The prospects for further research may include creations of models for the optimal control trajectories findings on conditions involving rates of the considered values varying and in probabilistic, stochastic, undetermined problem settings.

Keywords: hybrid function, multi-optional control, distribution density, optimal path, variational principle, optimal controlling surface, optional functions entropy.

NOMENCLATURE

y is an outlet value of the control system;
 t is a functions independent argument (time);
 k_0 is a coefficient characterizing the control influence effectiveness;
 u is a control function;
 k_f is a coefficient characterizing the disturbance influence effectiveness;
 f is a disturbance function;
 k_p is a coefficient of a governor augmentation; a function of time; a functions independent argument;
 ε is an error function;

x is a given action function (an inlet value of the control system);

L_ε is a rate of the losses stipulated by the error;

C_ε is a coefficient of the error;

L_{k_p} is a rate of the coefficient of the governor augmentation increase cost;

C_{k_p} is a coefficient of the governor augmentation coefficient;

n is a power index of the governor augmentation coefficient;

J is an objective functional of the total expenses related to the process of control optimization;

t_0 is an initial time of the process;

t_1 is a terminal time of the process;

F is an under-integral function of a functional;

k'_p is a first complete derivative of an unknown (free) function of the governor augmentation coefficient with respect to the independent variable (time);

k_p^* is an optimal function (extremal) delivering an optimal (minimal/maximal) value to an integral functional;

k_{p_0} is an initial value of the coefficient of the governor augmentation range;

k_{p_1} is a terminal value of the coefficient of the governor augmentation range;

Φ_h is an objective functional of a hybrid optional function distribution density;

h is a hybrid optional function distribution density;

β is an internal structural parameter of the system optimal behavior;

γ is an internal structural parameter for a normalizing condition;

$\Delta t \Delta k_p$ is a degree of accuracy at the hybrid optional function distribution density entropy determination;

H_h is an entropy of the hybrid optional function distribution density;

h^* is an extremal control surface;

h'_t is a first partial derivative of the hybrid optional function distribution density with respect to time (the first independent variable);

h'_{k_p} is a first partial derivative of the hybrid optional function distribution density with respect to the coefficient of the governor augmentation (the second independent variable).

INTRODUCTION

According to the contemporary progress in the development of the diagnosing and recognizing models synthesis it is still an actual scientific problem (task) in general to implement the modern achievements in the field of information technologies [1]. The importance of the issue lies in the plane of connections of the up to date technologies between themselves (a compatibility aspect) and putting them into practice.

Therefore scientific basis for the development of the presented theme will deal with modeling the optimal controlling process in regards to hybrid optional functions distributions densities taking into account the expenses related with the process.

Thus, this justifies the study of optional hybrid approaches combining different theoretical concepts and urgency of finding elements of generalization on the basis of the critical analysis and comparison with already known solutions for the synthesized models.

Therefore, the object of the presented study is the optimal control process which generates a problematic situation of the optimum definition, evaluation, and determination. And the subject of the study contained within the object is the

rough model of the phenomenon, and simplified dependence of optimal controlling trajectory upon the cost of control in an inertness-less linear controlling system equipped with a proportional governor.

The aim of the work is to build up an adequate model of the mentioned above process development. And the tasks needed to be solved to achieve this aim are formulated as follows: to develop methods for quantitative and qualitative evaluations of relationships between: 1) the system accuracy behavior, 2) optimal values obtained on the basis of an expenditures principle, and 3) optimal controlling trajectory determination on the basis of optional hybrid functions densities entropy paradigm; with further justification of the methods through computer simulation and calculation experimenting, as well as the obtained results interpretation and discussion.

1 PROBLEM STATEMENT

Suppose a linear inertness-less object of control with the time dependent outcome value: $y(t)$ is given. As for a linear control system the structure of $y(t)$ also depends upon the influences of control $u(t)$ and disturbance $f(t)$ and it is regularly represented with their sum: $y(t) = k_0 u(t) + k_f f(t)$, where k_0 , k_f – coefficients which characterize the effectiveness of the corresponding influence exerted upon the object. If the system is equipped with a proportional governor, which means $u(t) = k_p \varepsilon(t)$, where k_p – coefficient of the governor augmentation; $\varepsilon(t)$ – error value, then we come to the problem of the error minimization with the help of the k_p coefficient increase since $k_p \rightarrow \max$ provides $\varepsilon(t) \rightarrow \min$.

However, the value of the coefficient k_p is restricted. Therefore, through the prism of the system accuracy cost analysis the synthesized control trajectory will have some optimal value: $k_p(t) \rightarrow \text{opt}$ giving the minimal cost to the system accuracy.

In turn, the problem of the optimal function of $k_p^*(t)$ formation is to find an extremal of a functional: $\Phi_h \rightarrow \max$ through a hybrid optional function distribution density: $h^*(t, k_p)$ with taking into account the densities uncertainty measure (entropy): H_h and $k_p^*(t)$ delivers maximum to $h^*(t, k_p)$ as well as minimum to the cost to the system accuracy.

2 REVIEW OF THE LITERATURE

Theoretical core of the study is a synthesis of a hybrid approach on the basis of an optional uncertainty pattern. Mentioned in the introductory section of this paper problems of diagnostics and recognition models [1] has a variety in use.

It touches in actual fact the widest application of that kind of modeling, optimization, control, and all types of

artificial intelligence involvements, starting with that for aero-engines [2] and their maintenance and diagnosing [3, 4] respectively, as well as robust design works likewise [5] up to other aviation problems such as aviation noise and radio equipment qualities [6, 7] and finally ending with the social and economical ones [8–10].

Informativeness of diagnostic attributes let us say for their optional use like formulated in patent [11] nevertheless requires every time redefining the proposed criteria. There must be a certain general optional based parameter with a character of optimality in conditions of uncertainty of the considered options.

Uncertainty of some intrinsic value, preferences functions, in the view of their entropy is proposed in Subjective Analysis [12–15] and its applications [16–22]; and those analogues will be widely used in this work, however in the view of optional hybrid functions distributions densities entropy paradigm since the human-being influence is excluded from consideration rather the objectively existing matter is taking into account only.

The mathematical background for the presented paper is the cornerstones of fundamental theories such as [23–27] but not restricted just to those.

3 METHODS

Let us consider a linear inertness-less object of control. Supposedly, its behavior is described with the equation of [23, P. 162, (1)]:

$$y = k_0 u + k_f f, \quad (1)$$

where y – outlet value $y(t)$ changing in time t ; k_0 , k_f – coefficients which characterize the effectiveness of the corresponding influence exerted upon the object; u – control $u(t)$; f – disturbance $f(t)$.

The concept described with Eq. (1) implies simplifications that idealize a real object behavior.

If there is a proportional governor in the control system, then its behavior equation has the view of [23, P. 163, (2)]:

$$u = k_p \varepsilon, \quad (2)$$

where k_p is a coefficient of the governor augmentation; ε is an error value.

Traditionally, the accuracy of a control system is assessed with the value of, [23, pp. 160–165]:

$$\varepsilon(t) = x(t) - y(t), \quad (3)$$

where $x(t)$ is a given action which is predetermined by the task being solved.

The considered approach developed of the Eq. (1)–(3) yields the theoretical result represented in the following sequence of formulas.

From the relation (3) we can express the given predetermined value of $x(t)$ on the basis of the relation (4):

$$x(t) = y(t) + \varepsilon(t). \quad (4)$$

The error $\varepsilon(t)$, of the dependences (3), (4), in its turn from the interrelation (2), for the Eq. (4) will be expressed with the help of the equation (5):

$$\varepsilon(t) = \frac{u(t)}{k_p}, \quad (5)$$

where the control $u(t)$, from the formula (1), is written as the expression (6):

$$u(t) = \frac{y(t) - k_f f(t)}{k_0}. \quad (6)$$

Hence, on condition of the formula (6) the error $\varepsilon(t)$ from Eq. (5) becomes a functional dependence defined by the formula (7):

$$\varepsilon(t) = \frac{y(t) - k_f f(t)}{k_0 k_p}. \quad (7)$$

Thus, substituting the expression (7) value into the Eq. (4), one can obtain the interrelationship between the input – $x(t)$, output – $y(t)$ of the governed object, and the disturbance – $f(t)$ influencing the object in the view of the next formula (8):

$$x(t) = y(t) + \frac{y(t) - k_f f(t)}{k_0 k_p}. \quad (8)$$

Adding the right hand part of the Eq. (8) to common denominator it gets it the other notation (9):

$$x(t) = \frac{k_0 k_p y(t) + y(t) - k_f f(t)}{k_0 k_p}. \quad (9)$$

Developing the obvious transformations for Eq. (9) we obtain dependence (10):

$$k_0 k_p x(t) = (k_0 k_p + 1)y(t) - k_f f(t). \quad (10)$$

Taking into account that from formula (3), (4)

$$y(t) = x(t) - \varepsilon(t), \quad (11)$$

and substituting the value of the Eq. (11) for its value into the dependence (10) it yields the needed expression for the relation (11), between the error, inlet, and disturbance:

$$k_0 k_p x(t) = (k_0 k_p + 1)[x(t) - \varepsilon(t)] - k_f f(t). \quad (12)$$

After several simplest transformations of the Eq. (12) one can receive the intermediate expressions (13)–(15):

$$k_0 k_p x(t) = (k_0 k_p + 1)x(t) - (k_0 k_p + 1)\varepsilon(t) - k_f f(t), \quad (13)$$

$$(k_0 k_p + 1)\varepsilon(t) = (k_0 k_p + 1)x(t) - k_0 k_p x(t) - k_f f(t), \quad (14)$$

$$\varepsilon(t) = \frac{(k_0 k_p + 1)x(t) - k_0 k_p x(t) - k_f f(t)}{k_0 k_p + 1}. \quad (15)$$

And finally, from the Eq. (13)–(15), the workable view relationship with the explicitly expressed objective values will be the dependence (16):

$$\varepsilon(t) = x(t) \left(1 - \frac{k_0 k_p}{k_0 k_p + 1} \right) - \frac{k_f}{k_0 k_p + 1} f(t) = \varepsilon(t, k_p). \quad (16)$$

As far as we can judge from the relation (16) that increasing the coefficient $k_p(t)$ it is possible to decrease the value of the error $\varepsilon(t)$. In actual fact, the increase of the governor augmentation coefficient $k_p(t)$ cannot be unlimited or endless.

Thus, we come to the problem of the optimal value of the coefficient $k_p(t)$ of the control system. Application of an analogue to the Subjective Entropy Maximum Principle [12–22] helps us solving such a problem with respect to distributions that might be considered an elements approach in the given case study.

Let us apply an Expenditures Principle which assumes that the rate of the losses stipulated by the error of the control system is proportional to the absolute value of the error $\varepsilon(t, k_p(t))$, i.e.

$$L_\varepsilon(t, k_p(t)) = C_\varepsilon |\varepsilon(t, k_p(t))|, \quad (17)$$

where C_ε is a coefficient.

The cost of the coefficient $k_p(t)$ rate let be nonlinearly increasing, that is

$$L_{k_p}(k_p(t)) = C_{k_p} k_p^n(t), \quad (18)$$

where C_{k_p} is a coefficient; n is a power index.

Then, the total expenses related to the process of the control optimization will be found as the integral functional (19) of the sum of these two mentioned above components (17) and (18):

$$J[k_p(t)] = \int_{t_0}^{t_1} [L_\varepsilon(t, k_p(t)) + L_{k_p}(k_p(t))] dt = \int_{t_0}^{t_1} [C_\varepsilon |\varepsilon(t, k_p(t))| + C_{k_p} k_p^n(t)] dt. \quad (19)$$

From Eq. (16)

$$\varepsilon(t, k_p(t)) = \frac{x(t) - k_f f(t)}{k_0 k_p(t) + 1}. \quad (20)$$

Then the functional (19) with respect to Eq. (20) becomes

$$J[k_p(t)] = \int_{t_0}^{t_1} \left[C_\varepsilon \left| \frac{x(t) - k_f f(t)}{k_0 k_p(t) + 1} \right| + C_{k_p} k_p^n(t) \right] dt. \quad (21)$$

The functional (21) optimal value with taking into consideration the governor augmentation coefficient $k_p(t)$ will be found on the basis of the Euler-Lagrange equation (22):

$$\frac{\partial F(t, k_p(t))}{\partial k_p(t)} - \frac{d}{dt} \left(\frac{\partial F(t, k_p(t))}{\partial k'_p(t)} \right) = 0, \quad (22)$$

where $F(t, k_p(t))$ – under-integral function of the functional (19), (21), i.e.

$$F(t, k_p(t)) = C_\varepsilon \left| \frac{x(t) - k_f f(t)}{k_0 k_p(t) + 1} \right| + C_{k_p} k_p^n(t), \quad (23)$$

$k'_p(t)$ is a first complete derivative of the unknown (free) function of the governor augmentation coefficient $k_p(t)$ with respect to the independent variable (that is the variable of time t):

$$k'_p(t) = \frac{dk_p(t)}{dt}. \quad (24)$$

Since by assumption the under-integral function of the functional (19), (21) does not depend upon the first complete derivative of $k'_p(t)$ Eq. (24), that is $F(t, k_p(t))$ only, then for

the $\frac{\partial F(t, k_p(t))}{\partial k'_p(t)}$ member of the Euler-Lagrange Eq. (22) it

means

$$\frac{\partial F(t, k_p(t))}{\partial k'_p(t)} \equiv 0, \quad (25)$$

$$\frac{d}{dt} \left(\frac{\partial F(t, k_p(t))}{\partial k'_p(t)} \right) \equiv 0, \quad (26)$$

and for the entire Euler-Lagrange Eq. (22) we have one very important partial case of

$$\frac{\partial F(t, k_p(t))}{\partial k_p(t)} = 0. \quad (27)$$

The optimal function (extremal) of $k_p^*(t)$ delivering the optimal (minimal) value to the integral functional (19), (21) is obtained on the necessary condition of the functional extremum existence in the view of Eq. (27).

In order to shorten the notation starting from now we denote $k_p(t)$ simply as k_p remembering its functioning of independent variable (time t) value.

The Eq. (23) on condition of the Eq. (27) yields the dependence (28):

$$\begin{aligned} \frac{\partial F(t, k_p)}{\partial k_p} &= \frac{\partial}{\partial k_p} \left(C_\varepsilon |x(t) - k_f f(t)| \frac{1}{k_0 k_p + 1} + C_{k_p} k_p^n \right) = \\ &= C_\varepsilon |x(t) - k_f f(t)| \frac{\partial}{\partial k_p} \left(\frac{1}{k_0 k_p + 1} \right) + C_{k_p} \frac{\partial}{\partial k_p} (k_p^n) = \\ &= -\frac{C_\varepsilon |x(t) - k_f f(t)| k_0}{(k_0 k_p + 1)^2} + n C_{k_p} k_p^{n-1} = 0. \end{aligned} \quad (28)$$

From the expression (28) we get the relation (29):

$$nC_{k_p} k_p^{n-1} = \frac{C_\varepsilon |x(t) - k_f f(t)| k_0}{(k_0 k_p + 1)^2} = \frac{C_\varepsilon |x(t) - k_f f(t)| k_0}{(k_0 k_p)^2 + 2k_0 k_p + 1} \quad (29)$$

and finally the formula (30):

$$nC_{k_p} k_p^{n-1} [(k_0 k_p)^2 + 2k_0 k_p + 1] = C_\varepsilon |x(t) - k_f f(t)| k_0 \quad (30)$$

Or the relation (31):

$$k_p^{n-1} (k_0 k_p + 1)^2 = \frac{C_\varepsilon |x(t) - k_f f(t)| k_0}{nC_{k_p}} \quad (31)$$

For recursion or recurring or iteration it is interpreted with

$$k_p = n-1 \sqrt{\frac{C_\varepsilon |x(t) - k_f f(t)| k_0}{nC_{k_p} (k_0 k_p + 1)^2}} \quad (32)$$

For the relationships (29)–(32) root equation $k_p^*(t)$ determination we propose certain dependence defined from Eq. (31) with the formula (33):

$$k_p^{n-1} (k_0 k_p + 1)^2 - \frac{C_\varepsilon |x(t) - k_f f(t)| k_0}{nC_{k_p}} = 0 \quad (33)$$

Since it is difficult to find the explicit analytical function of the optimal controlling path determination, any of the numerical solutions in the view of the expressions (29)–(33) can be used as well as a table function definition.

Now we propose an optional approach to the stated problem solution. It implies considering the governor augmentation coefficient value k_p as an optional value having a possibility of changing in a certain range $[k_{p0} \dots k_{p1}]$. The process of the object control develops in some time diapason $[t_0 \dots t_1]$. The control system optimal dynamic characteristics here will also be the extremal of $k_p^*(t)$ (as that one yielded in the procedures of Eq. (1)–(33)) but for the considered case with taking into account the optional nature of k_p as well as a degree of uncertainty of specifically introduced Hybrid Functions. A presented study is a development of Subjective Analysis, Active System, Multi-alternativeness Concepts, Subjective Preferences Approach, and Subjective Entropy Paradigm, Subjective Entropy Maximum Principle considered in the sequence of publications [12–22].

Here we consider a hybrid functions apparatus equivalent to the preferences functions one; however the hybrid functions are an intrinsic property of the system, being pertained to the system itself on the contrary with the preferences functions pertaining to the subject (active element of the system, person responsible for making controlling decisions, individual). Thus, we remove from taking into account a human-being, considering technically

objectively existing optima instead although.

In such conceptual framework the objective functional is analogous with the one discussed in papers [19], [16], and dissertation [17, 18]:

$$\Phi_h = \int_{t_0}^{t_1} \int_{k_{p0}}^{k_{p1}} \left\{ -h(t, k_p) \ln h(t, k_p) - \beta h(t, k_p) [C_\varepsilon |\varepsilon(t, k_p)| + C_{k_p} k_p^n] dk_p \right\} dt + \left[\int_{t_0}^{t_1} \int_{k_{p0}}^{k_{p1}} h(t, k_p) dk_p dt - 1 \right] - \ln(\Delta t \Delta k_p) \quad (34)$$

where $h(t, k_p)$ is a hybrid optional function distribution density similar with the preference function distribution density [19], [16], and [17, 18]; β, γ are internal structural parameters of the system optimal behavior, likewise for the active element's psych [12–22] (endogenous parameter for the function of the optional effectiveness

$h(t, k_p) [C_\varepsilon |\varepsilon(t, k_p)| + C_{k_p} k_p^n]$ and uncertain Lagrange multiplier for the normalizing condition

$\int_{t_0}^{t_1} \int_{k_{p0}}^{k_{p1}} h(t, k_p) dk_p dt - 1$ respectively); $\Delta t \Delta k_p$ – degree of

accuracy at the hybrid optional function distribution density entropy (analogous to the subjective entropy of the preferences) determination.

Here in the functional (34) the first underintegral member is the entropy of the hybrid optional function distribution density $h(t, k_p)$:

$$H_h = \int_{t_0}^{t_1} \int_{k_{p0}}^{k_{p1}} \left\{ -h(t, k_p) \ln h(t, k_p) \right\} dk_p dt - \ln(\Delta t \Delta k_p) \quad (35)$$

conventionally without the logarithm of the degree of accuracy at the entropy determination $\Delta t \Delta k_p$.

Also, here, in functional (34), there applied the sign “minus” before the internal structural parameter of the system optimal behavior (likewise endogenous parameter of the active element's psych [12–22]) at the value of $\beta > 0$ since we guess it is a better option having the minimal value of the system effectiveness function – the sum of the rates (17) and (18), and which has to be found in case of the two independent variables although.

For obtaining an extremal surface of $h^*(t, k_p)$ in such a case we will need the Euler-Lagrange equation in the view of the applicable formula for the functional of (34) in the case of the two independent variables, [24]:

$$\frac{\partial F(t, k_p, h(t, k_p))}{\partial h(t, k_p)} - \frac{\partial}{\partial t} \left(\frac{\partial F(t, k_p, h(t, k_p))}{\partial h'_t(t, k_p)} \right) - \frac{\partial}{\partial k_p} \left(\frac{\partial F(t, k_p, h(t, k_p))}{\partial h'_{k_p}(t, k_p)} \right) = 0 \quad (36)$$

where now $F(t, k_p, h(t, k_p))$ – under-integral function of the functional (34), $h'_t(t, k_p)$ and $h'_{k_p}(t, k_p)$ – partial derivatives of the optional function with respect to the corresponding independent variables.

Since $F(t, k_p, h(t, k_p))$ of the functional (34) does not depend upon $h'_t(t, k_p)$ and $h'_{k_p}(t, k_p)$, then

$$\frac{\partial F(t, k_p, h(t, k_p))}{\partial h'_t(t, k_p)} = \frac{\partial F(t, k_p, h(t, k_p))}{\partial h'_{k_p}(t, k_p)} \equiv 0 \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial t} \left(\frac{\partial F(t, k_p, h(t, k_p))}{\partial h'_t(t, k_p)} \right) = \frac{\partial}{\partial k_p} \left(\frac{\partial F(t, k_p, h(t, k_p))}{\partial h'_{k_p}(t, k_p)} \right) \equiv 0. \quad (37)$$

And the conditions of the Eq. (36) and (37) applied to the functional (34) yield the necessary condition of the functional (34) extremum existence in the view of the formula (38):

$$\frac{\partial F(t, k_p, h(t, k_p))}{\partial h(t, k_p)} = 0. \quad (38)$$

On the basis of Eq. (38) from the functional (34) we get the relation (39):

$$-\ln h(t, k_p) - 1 - \beta [C_\varepsilon |\varepsilon(t, k_p)| + C_{k_p} k_p^n] + \gamma = 0. \quad (39)$$

From Eq. (39) we obtain the expression (40):

$$\ln h(t, k_p) = \gamma - 1 - \beta [C_\varepsilon |\varepsilon(t, k_p)| + C_{k_p} k_p^n],$$

$$h(t, k_p) = e^{\gamma - 1 - \beta [C_\varepsilon |\varepsilon(t, k_p)| + C_{k_p} k_p^n]}. \quad (40)$$

On the basis of the normalizing condition

$$\int_{t_0}^{t_1} \int_{k_{p0}}^{k_{p1}} h(t, k_p) dk_p dt = 1 = \int_{t_0}^{t_1} \int_{k_{p0}}^{k_{p1}} e^{\gamma - 1 - \beta [C_\varepsilon |\varepsilon(t, k_p)| + C_{k_p} k_p^n]} dk_p dt =$$

$$= e^{\gamma - 1} \int_{t_0}^{t_1} \int_{k_{p0}}^{k_{p1}} e^{-\beta [C_\varepsilon |\varepsilon(t, k_p)| + C_{k_p} k_p^n]} dk_p dt.$$

and

$$e^{\gamma - 1} = \frac{1}{\int_{t_0}^{t_1} \int_{k_{p0}}^{k_{p1}} e^{-\beta [C_\varepsilon |\varepsilon(t, k_p)| + C_{k_p} k_p^n]} dk_p dt}. \quad (41)$$

Substituting the result of (41) into the expression (40) we find the canonical distribution of the hybrid optional

function density as the extremal surface of $h^*(t, k_p)$, that delivers maximal value to the functional (34) and is in that sense the optimal controlling surface – the dependence (42):

$$h^*(t, k_p) = \frac{e^{-\beta [C_\varepsilon |\varepsilon(t, k_p)| + C_{k_p} k_p^n]}}{\int_{t_0}^{t_1} \left(\int_{k_{p0}}^{k_{p1}} e^{-\beta [C_\varepsilon |\varepsilon(t, k_p)| + C_{k_p} k_p^n]} dk_p \right) dt}. \quad (42)$$

4 EXPERIMENTS

Calculation experiments illustrate the theoretical speculations (1)–(42) of the above sections and subsections of the presented researches.

The numerical simulation has been performed for both one-dimensional and two-dimensional modeling cases. The accepted conditions were as follows:

$$x(t) = f(t) = A_x(t) \sin[\omega_x(t)t + T_{x_0}], \quad T_{x_0} = 3, \quad T_{a_0} = 2,$$

$$A_x(t) = a_x(t) \sin[\omega_a(t)t + T_{a_0}],$$

$$\omega_x(t) = 8 \cdot 10^{-5} t \cos[\omega_{a_1} t + \omega_{a_0}],$$

$$a_x(t) = 1 \cdot 10^{-3} t \sin(a_{x_1} t + a_{x_0}), \quad \omega_a(t) = \omega_x(t),$$

$$a_{x_0} = 4 \cdot 10^{-2}, \quad a_{x_1} = \omega_{a_1} = 2 \cdot 10^{-2}, \quad \omega_{a_0} = 4 \cdot 10^{-3},$$

$$k_f = 9, \quad k_0 = 5 \cdot 10^{-3}, \quad C_\varepsilon = 1, \quad C_{k_p} = 10^{-6}, \quad n = 2, \quad t_0 = 0,$$

$$t_1 = 5 \cdot 10^2, \quad k_{p_0} = 0, \quad k_{p_1} = 1 \cdot 10^3, \quad \beta = 4.5.$$

The obtained results of the mathematical modeling are shown in Fig. 1.

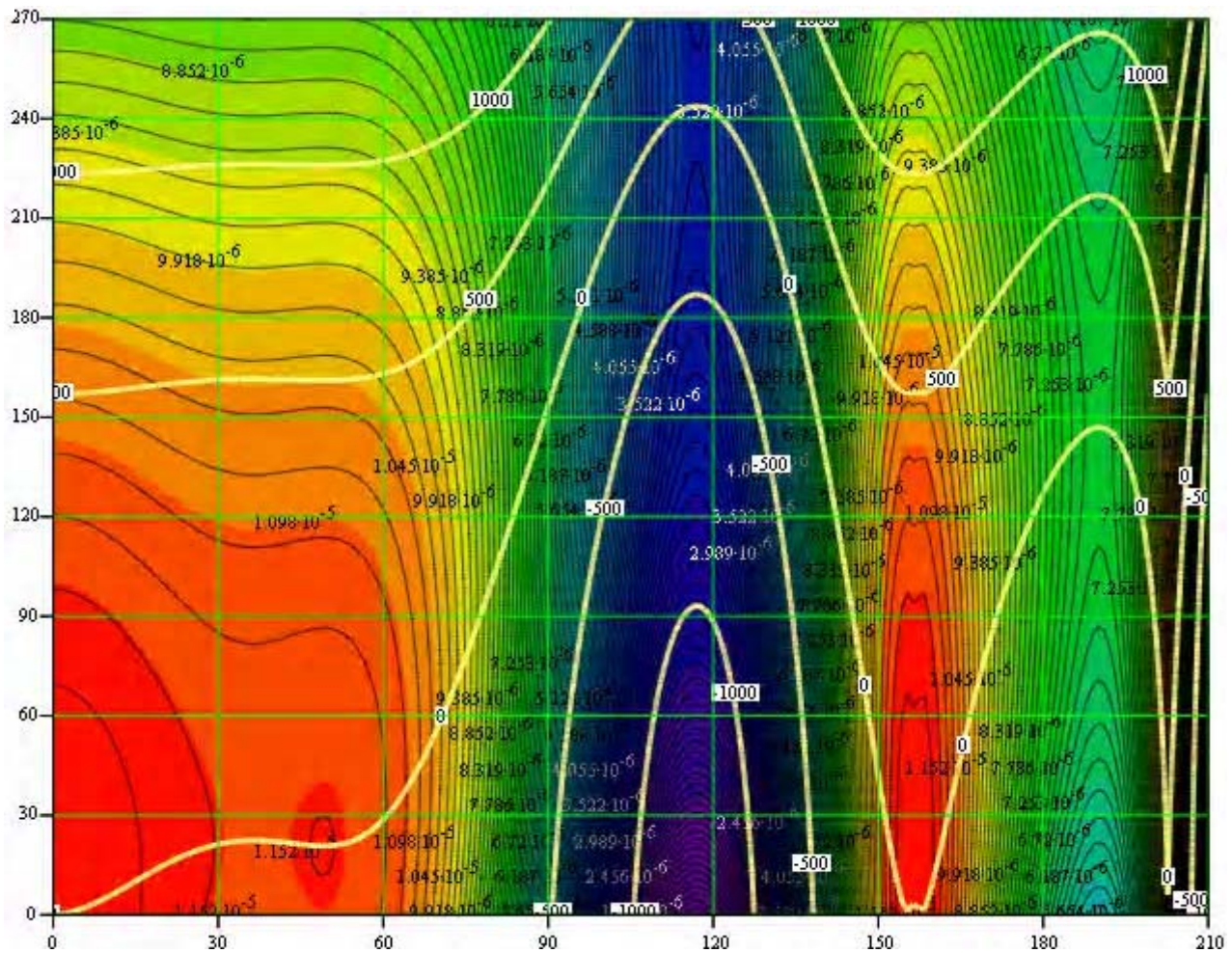
The hybrid optional function distribution density as the extremal surface of $h^*(t, k_p)$, obtained by the formula (42),

depicted as "h" in Fig. 1, is shown in conjunction with the corresponding surface of the sum of the rates (17) and (18), in the view of the function of the two independent variables of t and k_p although, represented with the designation of "Sum" in the view of contour plots for the conveniences.

Also in Fig. 1 the surface indicated as "Z" is shown. It illustrates the contour lines obtained from Eq. (33) in yellow color. The contour lines shown are marked for the paces of: "–1000", "–500", "0", "500", "1000".

The fragment portrayed in Fig. 1 is represented for the time zone at the abscissa axis $0 \leq t \leq 210$ and governor augmentation coefficient range at the ordinate axis $0 \leq k_p \leq 270$.

The curve marked "0" (see Fig. 1) obtained as the root equation $k_p^*(t)$ of the expression (33) should be considered as the optimal controlling path.



h, Sum, Z

Figure 1 – Optimal value of the governor augmentation coefficient $k_p^*(t)$ with respect to the objective effectiveness function

$$C_\varepsilon |\varepsilon(t, k_p)| + C_{k_p} k_p^n \text{ and hybrid optional function distribution density } h^*(t, k_p)$$

6 DISCUSSION

The concepts described with the formulae of (1)–(42) yield the optimal value for the governor augmentation coefficient (see Fig. 1):

$$k_p^*(t) = \arg \max h^*(t, k_p), \quad (43)$$

which at the same time is

$$k_p^*(t) = \arg \min [C_\varepsilon |\varepsilon(t, k_p)| + C_{k_p} k_p^n], \quad (44)$$

that is the maximal value of the hybrid optional function distribution density $h^*(t, k_p)$ (42) is ensured with the value of $k_p^*(t)$, as the continuous optional value, which guarantees the minimal value of the objective effectiveness function $C_\varepsilon |\varepsilon(t, k_p)| + C_{k_p} k_p^n$.

Besides, the optimal value of the governor augmentation coefficient $k_p^*(t)$, the root equation of the expression (33),

is found in result of optimization of the objective functional (19) with taking into consideration the expenses (17) and (18) related to the error of the controlling actions and increase of the augmentation coefficient.

Furthermore, the optional hybrid density $h^*(t, k_p)$, given with the formula (42), itself in its turn is the optimal argument that maximizes the synthesized objective functional (34),

$$h^*(t, k_p) = \arg \max \Phi_h, \quad (45)$$

taking into account the uncertainty (that is represented with the entropy member, the formula (35) in the functional) of the normalized optional hybrid value $h(t, k_p)$.

All this allows treating the optional hybrid density as the optimal controlling surface with the relativity of its magnitude.

The results described with the formulae (43)–(45) continue and generalize research initiated in works [12–22], especially [19], [16], as well as [17, 18].

Critical comparison of the achieved results with the analogues [23] shows advantages of the proposed method as it takes into consideration the cost of the controlling system accuracy and centers a local optimum whereas without such assessments there is none.

Moreover, control in conditions of uncertainty in the given problem setting allows making allowance for the uncertainty of the hybrid optional functions distribution densities with respect to the objective effectiveness functions, which significantly differs from results discussed in monograph [25].

We should also note that the presented method, developed on the basis of variational principles [24] in application to the subjective analysis theory [11–14] actually dealing with the given sets of both discrete and continuous alternatives as well as uncertainty entropy measures for the system of the two independent variables [27], in fact does not have anything in common with the active system rather than objectively existing intrinsic properties of the controlled system.

It definitely has to have development in terms of mass service systems theory [26] in the direction of the entropy paradigm research.

CONCLUSIONS

The urgent problem of a mathematical model synthesis for the augmentation coefficient optimal value of a proportional governor included into an inertness-less linear object control system determination is solved.

The method of hybrid optional function distribution density entropy is firstly proposed. The discovered value of the hybrid optional function distribution density has a property of, and allows determining, an optimal value with respect to the synthesized objective functional concerning the uncertainty and normalization of such an option.

The practical significance of the obtained results is that the hybrid optional density delivering the maximal value to the synthesized functional has its own maximum that provides minimum in regards with the integrated cost of the controlling process. That must be taken as the optimal controlling path.

ACKNOWLEDGEMENTS

The work is supported by the state budget scientific research project of National Aviation University, Kyiv, “Synthesis of optimal control algorithms for a complex active technical object operation” (state registration number 0116U004637).

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Article was submitted 12.06.2017.
After revision 12.08.2017.

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ВИЗНАЧЕННЯ ОПТИМАЛЬНОЇ ТРАЕКТОРІЇ КЕРУВАННЯ ЗА ДОПОМОГОЮ РОЗПОДІЛІВ ГІБРИДНИХ ОПЦІЙНИХ ФУНКЦІЙ

Актуальність. Вирішено завдання визначення оптимального значення коефіцієнта підсилення пропорційного регулятора, включеного до системи керування безінерційного лінійного об'єкта, на основі синтезованої моделі.

Мета роботи – створення методу для відшукування розв'язку за наявності проблемної ситуації пов'язаної із визначенням та оцінкою оптимуму в системі керування.

Метод. Запропоновано грубу модель явища, та спрощену залежність оптимальної траєкторії керування від вартості, керування в безінерційній лінійній системі керування оснащеної пропорційним регулятором. Точність поведінки досліджуваного лінійного об'єкта керування обрано у даному розгляді у якості початкової цільової величини, що потребує мінімізації. Запропоновано метод побудови моделі з урахуванням затратного принципу, який забезпечує розрахунок вартості контрольованого процесу, що дозволяє знайти оптимальне керує значення на мульти-опційній основі. Застосовано деякий аналог принципу максимуму суб'єктивної ентропії із суб'єктивного аналізу із метою отримання специфічних оптимальних розподілів для цільової величини взятої у вигляді складеного функціоналу. Метод екстремізації ступеню невизначеності опцій удосконалено за допомогою введення неперервної опційної величини, що дозволяє сформувати щільність розподілу тієї величини. Побудовано опційну синтезовану модель процесу керування.

Результати. Розроблені теоретичні моделі дозволяють отримати, та були впроваджені при її знаходженні, гібридну опційну щільність у якості оптимального розв'язку варіаційної задачі із двома незалежними змінними, чие максимальне значення є шуканою оптимальною траєкторією керування, яка доставляє мінімум інтегральним витратам притаманним даному процесові.

Висновки. Проведено чисельні експерименти з дослідження запропонованих методів у даній задачі оптимізації. У результаті тих експериментів виявлені залежності є обґрунтованими, їхнє застосування на практиці дозволяє виконувати, та є рекомендованим за необхідності визначати, оптимальне керування в описаних системах. Перспективи подальших досліджень можуть полягати у створенні моделей для визначення оптимальних траєкторій керування, в умовах які передбачають швидкості змін величин, що розглядаються, а також у імовірнісній, стохастичній, недетермінованій постановці.

Ключові слова: гібридна функція, мульти-опційніне керування, щільність розподілу, оптимальна траєкторія, варіаційний принцип, оптимальна керуюча поверхня, ентропія опційних функцій.

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ОПРЕДЕЛЕНИЕ ОПТИМАЛЬНОЙ ТРАЕКТОРИИ УПРАВЛЕНИЯ С ПОМОЩЬЮ РАСПРЕДЕЛЕНИЙ ГИБРИДНЫХ ОПЦИОННЫХ ФУНКЦИЙ

Актуальность. Решена задача определения оптимального значения коэффициента усиления пропорционального регулятора, включенного в систему управления безынерционного линейного объекта, на основе синтезированной модели.

Цель работы – создание метода для отыскания решения при наличии проблемной ситуации связанной с определением и оценкой оптимума в системе управления.

Метод. Предложены грубая модель явления, и упрощенная зависимость оптимальной траектории управления от стоимости, управления в безынерционной линейной системе управления оснащенной пропорциональным регулятором. Точность поведения исследуемого линейного объекта управления выбрана в данном рассмотрении в качестве начальной целевой величины нуждающейся в минимизации. Предложен метод построения модели с учетом затратного принципа, который обеспечивает расчет стоимости контролируемого процесса, что позволяет найти оптимальное управляющее значение на мульти-опционной основе. Применен некоторый аналог принципа максимума субъективной энтропии из субъективного анализа с целью получения специфических оптимальных распределений для целевой величины взятой в виде составленного функционала. Метод экстремизации степени неопределенности опций усовершенствован посредством введения непрерывной опционной величины, что позволяет сформировать плотность распределения этой величины. Построена опционная синтезированная модель процесса управления.

Результаты. Разработанные теоретические модели позволяют получить, и были внедрены при ее нахождении, гибридную опционную плотность в качестве оптимального решения вариационной задачи с двумя независимыми переменными, чье максимальное значение является искомым оптимальной траекторией управления, доставляющей минимум интегральным расходам присущим данному процессу.

Выводы. Проведены численные эксперименты по исследованию предложенных методов в данной задаче оптимизации. В результате этих экспериментов выявленные зависимости являются обоснованными, их применение на практике позволяет выполнять, и является рекомендуемым при необходимости определять, оптимальное управление в описанных системах. Перспективы дальнейших исследований могут заключаться в создании моделей для определения оптимальных траекторий управления, в условиях предусматривающих скорости изменений рассматриваемых величин, а также в вероятностной, стохастической, недетерминированной постановке.

Ключевые слова: гибридная функция, мульти-опционное управление, плотность распределения, оптимальная траектория, вариационный принцип, оптимальная управляющая поверхность, энтропия опционных функций.

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