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ON THE ANALYTICAL SOLUTION OF A VOLTERRA INTEGRAL EQUATION FOR INVESTIGATION OF FRACTAL PROCESSES

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ABSTRACT

Context. We consider a Volterra integral equation of the first kind which may be applied to the data filtration and forecast of fractal random processes, for example, in information-telecommunication systems and in control of complex technological processes.

Objective. The aim of the work is to obtain an exact analytical solution to a Volterra integral equation of the first kind. The kernel of the corresponding integral equation is the correlation function of a fractal random process with a power-law structure function.

Method. The Volterra integral equation of the first kind is solved with the help of the standard Laplace transform method. The inverse Laplace transform leads to the calculation of the line integral of the function of complex variable. This integral is calculated as a sum of a residue part and integrals over the banks of cut. The corresponding integrals are obtained on the basis of the known expansions of special functions.

Results. We obtained an exact analytical solution of the Volterra integral equation the kernel of which is the correlation function of a fractal random process. The paper is based on a model where the structure function of the corresponding process is a power-law function. It is shown that the part of the solution that does not contain delta-function is convergent at any point if the Hurst exponent is larger than 0.5, i.e. if the process has fractal properties. It is shown that the obtained solution is a real-valued function. The obtained solution is verified numerically; it is also shown that our solution gives the correct asymptotic behavior. Although the solution contains an exponentially growing function of time, at large times the integral of the obtained solution asymptotically behaves as a power-law function.

Conclusions. It is important to stress that we obtained an exact solution of the Volterra integral equation under consideration rather than an approximate one. The obtained solution may be applied to the data filtration and forecast of fractal random processes. As is known, fractal processes take place in a huge variety of different systems, so the results of this paper may have a wide field of application.

KEYWORDS: Volterra equation of the first kind, Hurst exponent, Laplace transform, fractal process, exact analytical solution.

NOMENCLATURE

$c(t)$ is a structure function of the fractal random process;

$h(t,k)$ is a unknown function for which the solution of the integral equation is obtained, $t \ge 0$;

H is a Hurst exponent;

i is a complex unity;

$R(t)$ is a correlation function of the fractal random process;

$x(t)$ is a fractal random process under consideration;

$\Gamma(\alpha, z)$ is a incomplete Gamma function;

$\Gamma(z)$ is a Gamma function;

σ^2 is a process variance;

$\langle a(t) \rangle_t$ is a time average of a random process $a(t)$;

$H(p,k)$ is a Laplace transform of the function $h(t,k)$;

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$r(p,k)$ is a Laplace transform of the function $R(t+k)$;

$r(p)$ is a Laplace transform of the function $R(t)$;

${}_1F_1(\alpha,\beta,z)$ is a confluent hypergeometric function;

$B(\mu,\nu)$ is a Beta-function.

INTRODUCTION

This paper is devoted to the obtaining of an analytical solution to a Volterra equation of the first kind which may be applied to the data filtration and forecast of fractal random processes. The kernel of the corresponding integral equation is the correlation function of a fractal random process with a power-law structure function.

The model of the power-law structure function is a very popular model for the description of fractal processes. For example, it is used in the description of plasma fluctuations [1], in the description of the financial market data on the basis of the statistical physics methods [2–4], etc.

Self-similar processes take place in a huge variety of different systems: industry applications, control systems (see, for example, [5, 6]), information-telecommunication systems, financial markets, physical systems (Brownian motion, non-equilibrium fluctuations, etc.), geophysical time series, etc., see [7] and references therein.

In this paper we consider only continuous random processes. The problem of the solution of the Volterra integral equation under consideration was discussed in [8] where this problem was investigated in the framework of the Kolmogorov-Wiener filter. We should stress that, to obtain the weight function and the output of the Kolmogorov-Wiener filter, a Fredholm integral equation of the first kind should be solved rather than the Voterra one (see, for example, [9]). But the Volterra integral equation is of mathematical interest by itself. As is also known, the Volterra integral equation is a special case of the Fredholm integral equation, so it may be applied to practical investigations of fractal processes.

In this paper the idea of the solution of the Volterra integral equation is similar to that of [8], but the results of paper [8] should be refined in some places, see the corresponding discussion in Sec. 2.

The object of study is the Volterra integral equation of the first kind, the kernel of which is the correlation function of a fractal random process with a power-law structure function.

The subject of study is the analytic solution of the system under consideration.

The aim of the work is to obtain an exact analytical solution to the integral equation under consideration and to investigate its asymptotic behavior.

1 PROBLEM STATEMENT

We consider the following Volterra integral equation of the first kind

$$R(T+k)=\int_0^T d\tau h(\tau,k)R(T-\tau), \qquad (1)$$

where $k<T$ is a finite positive constant,

$$R(t)=\sigma^2-\frac{\alpha}{2}t^{2H}, \qquad (2)$$

and $h(\tau,k)$ is the unknown function. The problem is to obtain an analytical solution to eq. (1).

2 REVIEW OF THE LITERATURE

The models with a power-law structure function are widely used to describe fractal processes (see, for example, [1–4]). Fractal processes are widely used in investigations of different systems (see [5–7] and references therein).

In paper [8] a continuous random process $x(t)$ is given for $t\in[0,T]$. The process is assumed to be a stationary and ergodic one. The structure function $c(t)$ is assumed to be a power-law function:

$$c(\tau)\equiv\left\langle\left(x(t)-x(t-\tau)\right)^2\right\rangle_t=\alpha\cdot\tau^{2H}. \qquad (3)$$

where α is a positive constant, and H is the Hurst exponent.

In the model (3) the corresponding correlation function is

$$R(\tau)\equiv\left\langle\left(x(t+\tau)-\langle x(t)\rangle_t\right)\left(x(t)-\langle x(t)\rangle_t\right)\right\rangle_t = \\ =\sigma^2-\frac{\alpha}{2}\tau^{2H}. \qquad (4)$$

The Voterra integral equation of the first kind (1) is considered in [8] in the framework of the Kolmogorov–Wiener filter. Of course, it should be stressed that, in order to obtain the Kolmogorov–Wiener filter output, the Fredholm integral equation should be solved rather than the Volterra one. Nevertheless, the Volterra integral equation discussed in [8] is of mathematical interest by itself. The Volterra integral equation is also a special case of the Fredholm integral equation, so it may be applied to data filtration and forecast in some cases (maybe even not necessarily in the framework of the Kolmogorov-Wiener problem).

The problem of the solution of the integral equation under consideration is investigated in [8] with the help of the standard Laplace transform method [10]. The authors of [8] carefully divided the solution into two parts, one of which contains the Dirac delta-function. However, the results of paper [8] should be significantly refined. First of all, eq. (19) in [8] contains a complex function as a result because the incomplete Gamma-function $\Gamma(2H+1,-\lambda x)$ is complex-valued (see eq. (19) in [8]). Besides, a pole residue is not taken into account in [8] either.

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In this paper the integral equation (1) is analytically solved and the results of paper [8] are refined.

3 MATERIALS AND METHODS

Let us introduce the following Laplace transforms:

$$r(p,k)=\int_0^\infty dTR(T+k)e^{-p(T+k)},$$
$$r(p)=\int_0^\infty dTR(T)e^{-pT}, \tag{5}$$
$$H(p,k)=\int_0^\infty dTh(T,k)e^{-pT}.$$

Substituting $\xi=T-\tau$ into the right-hand side of (1), multiplying the both sides of (1) by $\exp(-p(t+k))$ and taking the integral over T, with account for (5) we obtain

$$\int_0^\infty dT\int_0^T d\xi e^{-p(T+k)}h(T-\xi,k)R(\xi)=r(p,k). \tag{6}$$

Multiplying the integrand on the left-hand side of (6) by $\exp(-p\xi)\exp(p\xi)$ and substituting $x=T-\xi$, $y=\xi$ into (6), with account for (5) we obtain

$$e^{-pk}H(p,k)r(p)=r(p,k), \tag{7}$$

which with account for (5) leads to

$$H(p,k)=\frac{\int_0^\infty dTR(T+k)e^{-pT}}{\int_0^\infty dTR(T)e^{-pT}}. \tag{8}$$

The standard definitions and tabulated integrals for The Gamma and incomplete Gamma functions are [11]:

$$\Gamma(\alpha,x)=\int_x^\infty dte^{-t}t^{\alpha-1},\quad \Gamma(\alpha)=\int_0^\infty dte^{-t}t^{\alpha-1},$$
$$\int_u^\infty dxx^{\nu-1}e^{-\mu x}=\mu^{-\nu}\Gamma(\nu,\mu u), \tag{9}$$
$$\int_0^\infty dxx^{\nu-1}e^{-\mu x}=\mu^{-\nu}\Gamma(\nu).$$

On the basis of (9) and (2) the integrals in the numerator and the denominator of (8) are calculated:

$$\int_0^\infty dTR(T)e^{-pT}=\frac{\sigma^2}{p}-\frac{\alpha\Gamma(2H+1)}{2p^{2H+1}}, \tag{10}$$

$$\int_0^\infty dTR(T+k)e^{-pT}=\frac{\sigma^2}{p}-\frac{\alpha e^{pk}\Gamma(2H+1,pk)}{2p^{2H+1}}.$$

With account for (10) and (8) the following expression for $H(p,k)$ can be obtained:

$$H(p,k)=\frac{2p^{2H}\sigma^2-\alpha e^{pk}\Gamma(2H+1,pk)}{2p^{2H}\sigma^2-\alpha\Gamma(2H+1)}. \tag{11}$$

Let us investigate the behavior of $H(p,k)$ when $p\to\infty$. As is known [11], if $x\to\infty$, then $\Gamma(\alpha,x)$ can be represented as a series:

$$\Gamma(\alpha,x)\big|_{x\to\infty}=x^{\alpha-1}e^{-x}\sum_{m=0}^\infty\frac{(-1)^m\Gamma(1-\alpha+m)}{x^m\Gamma(1-\alpha)}. \tag{12}$$

On the basis of (12) and (11) we obtain

$$H(p,k)\big|_{p\to\infty}=1-\frac{\alpha k^{2H}}{2\sigma^2}+O(p^{-1}). \tag{13}$$

According to (13), let us split (11) into two parts:

$$H(p,k)=1-\frac{\alpha k^{2H}}{2\sigma^2}+H'(p,k),\quad H'(p,k)\big|_{p\to\infty}=0. \tag{14}$$

As is known [10], the inverse Laplace transform can be calculated as

$$h(t,k)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}dpH(p,k)e^{pt}=\left(1-\frac{\alpha k^{2H}}{2\sigma^2}\right)\delta(t)+h'(t,k) \tag{15}$$

(here we use the fact that the inverse Laplace transform of a constant is the delta-function).

The function $h'(t,k)$ in (15) is the inverse Laplace transform of the function $H'(p,k)$:

$$h'(t,k)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}dpH'(p,k)e^{pt} \tag{16}$$

and in what follows we calculate it. It should be stressed that up to this point all the results coincide with [8], but the following result for $h'(t,k)$ significantly differs from [8].

The singular points of the function $H'(p,k)$ are the branch point $p=0$ and the poles. The function $H'(p,k)$

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satisfies the conditions of the Jordan lemma (see (13)), so the integral (16) is

$$h'(t,k) = I(t,k) + J(t,k), \quad (17)$$

where $I(t,k)$ is the sum of the integrals over the banks of cut and $J(t,k)$ is the pole residue part (see, for example, [12]).

The following banks of cut should be chosen: $p = xe^{i\pi}$ and $p = xe^{-i\pi}$, so

$$\begin{gathered} I(t,k) = I_1(t,k) + I_2(t,k), \\ I_1(t,k) = \frac{e^{i\pi}}{2\pi i}\int_0^\infty dxH'\left(xe^{i\pi},k\right)\exp\left(xe^{i\pi}t\right), \\ I_2(t,k) = \frac{e^{-i\pi}}{2\pi i}\int_\infty^0 dxH'\left(xe^{-i\pi},k\right)\exp\left(xe^{-i\pi}t\right). \end{gathered} \quad (18)$$

As can be seen from (14) and (11), the function $H'(p,k)$ contains the functions p^{2H}, $\exp(pk)$ and $\Gamma(2H+1, pk)$. Obviously,

$$\begin{gathered} e^{\pm i\pi} = -1, \ \exp\left(xe^{\pm i\pi}t\right) = \exp(-xt), \\ \left(xe^{\pm i\pi}\right)^{2H} = x^{2H}\left(\cos(2\pi H) \pm i\sin(2\pi H)\right). \end{gathered} \quad (19)$$

As is known [11], the function $\Gamma(2H+1, pk)$ can be expanded into a series:

$$\Gamma(\alpha, x) = \Gamma(\alpha) - \sum_{n=0}^{\infty}\frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)}. \quad (20)$$

With account for (20) and (19) one can obtain

$$\begin{gathered} \operatorname{Re}\left(\Gamma\left(2H+1, kxe^{i\pi}\right)\right) = \Gamma(2H+1) + \\ +(kx)^{2H}\cos(2\pi H)\sum_{n=0}^{\infty}\frac{(kx)^{1+n}}{n!(2H+1+n)}; \\ \operatorname{Im}\left(\Gamma\left(2H+1, kxe^{i\pi}\right)\right) = \\ = (kx)^{2H}\sin(2\pi H)\sum_{n=0}^{\infty}\frac{(kx)^{1+n}}{n!(2H+1+n)}, \\ \operatorname{Re}\left(\Gamma\left(2H+1, kxe^{-i\pi}\right)\right) = \operatorname{Re}\left(\Gamma\left(2H+1, kxe^{i\pi}\right)\right), \\ \operatorname{Im}\left(\Gamma\left(2H+1, kxe^{-i\pi}\right)\right) = -\operatorname{Im}\left(\Gamma\left(2H+1, kxe^{i\pi}\right)\right). \end{gathered} \quad (21)$$

It should be noticed that we consider processes with fractal properties, i.e. we consider cases where $H \in (0.5;1)$. In this range of parameters we have $\sin(2\pi H) < 0$, $\operatorname{Im}(\Gamma(2H+1, kxe^{-i\pi})) > 0$. On the basis of (18)–(21) the following result for $I(t,k)$ is obtained:

$$\begin{gathered} I(t,k) = \frac{1}{\pi}\int_0^\infty dxe^{-tx} f(x,k), \\ f(x,k) = \frac{\Psi(x)\Theta(x,k) - \Lambda(x)\Omega(x,k)}{\Lambda^2(x) + \Psi^2(x)}, \\ \Theta(x,k) = 2\sigma^2 x^{2H}\cos(2\pi H) - \alpha e^{-kx}A(x,k), \\ \Omega(x,k) = 2\sigma^2 x^{2H}\sin(2\pi H) + \alpha e^{-kx}B(x,k), \\ \Lambda(x) = 2\sigma^2 x^{2H}\cos(2\pi H) - \alpha\Gamma(2H+1), \\ \Psi(x) = 2\sigma^2 x^{2H}\sin(2\pi H), \\ A(x,k) = \operatorname{Re}\left(\Gamma\left(2H+1, xke^{-i\pi}\right)\right), \\ B(x,k) = \operatorname{Im}\left(\Gamma\left(2H+1, xke^{-i\pi}\right)\right) > 0. \end{gathered} \quad (22)$$

We should stress that in contrast to [8] our result (22) is a real-valued function. Let us investigate the convergence of the integral in (22). On the basis of (22), (20) and the property $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ one can obtain

$$\begin{gathered} f(x,k)\big|_{x\to\infty} = \frac{\alpha\Gamma(2H+1)\sin(2\pi H)}{4\sigma^4}\times \\ \times\left(2\sigma^2 - \alpha k^{2H}\right)\cdot x^{-2H} + O\left(x^{-2H-1}\right) \end{gathered} \quad (23)$$

so $f(x,k)\big|_{x\to\infty} \sim x^{-2H}$ from which it follows that $I(0,k)$ is convergent if $H \in (0.5;1)$. Obviously, $I(t,k)$ is convergent for $t > 0$. So $I(t,k)$ is convergent for any $t \geq 0$ if the process has fractal properties.

Let us calculate the poles of the function $H'(p,k)e^{pt}$. Obviously, to calculate the poles, we should equate the denominator of (11) to zero because of (14). The solutions of the corresponding equations are

$$z = \left(\frac{\alpha}{2\sigma^2}\Gamma(2H+1)\right)^{\frac{1}{2H}} e^{i\frac{\pi n}{H}}, \ n \in \mathbb{Z}. \quad (24)$$

According to [12], only the poles with $\arg(z) \in [-\pi, \pi]$ contribute to $J(t,k)$ in (17). We consider the case where $H \in (0.5;1)$, so from (24) we can see that the only pole that contributes to $J(t,k)$ is

$$p_0 = \left(\frac{\alpha}{2\sigma^2}\Gamma(2H+1)\right)^{\frac{1}{2H}}. \quad (25)$$

Let us investigate the function $H'(p,k)e^{pt}$ in the vicinity of the point $p = p_0$. Let us introduce the

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parameter $\xi = p - p_0$. As is known [11], if $|y|<|x|$, then the following expansion is true:

$$\Gamma(\alpha, x+y) = \Gamma(\alpha, x) - \\ -e^{-x} x^{\alpha-1} \sum_{m=0}^{\infty} \left[\frac{(-1)^m \Gamma(1-\alpha+m)}{x^m \Gamma(1-\alpha)} \left(1 - e^{-y} \sum_{l=0}^{m} \frac{y^l}{l!} \right) \right]. \quad (26)$$

On the basis of (11), (14), (25) and (26) one can obtain that in the vicinity of the point $p = p_0$ (i.e. in the vicinity of the point $\xi = 0$)

$$H'(p,k)e^{pt} = H'(p_0+\xi,k)e^{(p_0+\xi)t} = \\ = p_0 e^{p_0 t} \frac{\Gamma(2H+1) - e^{p_0 k}\Gamma(2H+1, p_0 k)}{2H\Gamma(2H+1)} \xi^{-1} + \\ +O(\xi^0) = \frac{\Gamma(2H+1) - e^{p_0 k}\Gamma(2H+1, p_0 k)}{2H\Gamma(2H+1)} \times \\ \times p_0 e^{p_0 t} (p-p_0)^{-1} + O\left((p-p_0)^0\right). \quad (27)$$

As can be seen from (27), the expansion of $H'(p,k)e^{pt}$ into a Laurent series of $p - p_0$ begins with the minus first term, so p_0 is a simple pole and

$$J(t,k) = \operatorname*{Res}_{p=p_0} H'(p,k)e^{pt} = e^{p_0 t} p_0 \times \\ \times \frac{\Gamma(2H+1) - e^{p_0 k}\Gamma(2H+1, p_0 k)}{2H\Gamma(2H+1)}, \quad (28)$$

it should be noticed that $J(t,k)$ is not taken into account in [8].

So, the following solution is obtained:

$$h(t,k) = \left(1 - \frac{\alpha k^{2H}}{2\sigma^2}\right)\delta(t) + I(t,k) + J(t,k), \quad (29)$$

where the explicit expressions for $I(t,k)$ and $J(t,k)$ are given in (22) and (28); the expression for p_0 from (28) is given in (25).

4 EXPERIMENTS

Numerical calculations for some parameters are made in order to verify the solution (29). The integral

$$\int_0^T d\tau h(\tau,k) R(T-\tau) = \left(1 - \frac{\alpha k^{2H}}{2\sigma^2}\right) R(T) + \\ + \int_0^T d\tau I(\tau,k) R(T-\tau) + \int_0^T d\tau J(\tau,k) R(T-\tau) \quad (30)$$

is compared to $R(t+k)$, i.e. the right-hand side of (1) is compared to the left-hand side of (1). The calculations are made on the basis of Mathcad 14 package.

The incomplete Gamma function, which is a built-in Mathcad function, is not defined for a negative second argument. So the functions $A(x,k)$ and $B(x,k)$ in (22) are introduced as

$$\Gamma(2H+1, z) = \int_0^{\infty} dx e^{-x} x^{2H} + \int_z^0 dx e^{-x} x^{2H}, \\ A(x,k) = \mathrm{Re}\left(\Gamma(2H+1, -xk)\right), \\ B(x,k) = -\mathrm{Im}\left(\Gamma(2H+1, -xk)\right), \quad (31)$$

the sign "–" in $B(x,k)$ in (31) is due to the fact that Mathcad interprets $\Gamma(2H+1, -xk)$ from (31) as $\Gamma(2H+1, xke^{i\pi})$; see (21).

We should notice that Mathcad fails to calculate the function $I(t,k)$ as the integral from 0 to ∞, so $I(t,k)$ is treated as

$$I(t,k) \approx \frac{1}{\pi} \int_0^{233} dx e^{-tx} f(x,k) + \frac{1}{\pi} \int_{233}^{+\infty} dx e^{-tx} \frac{C}{x^{2H}}, \\ C = \frac{\alpha\Gamma(2H+1)\sin(2\pi H)}{4\sigma^4}\left(2\sigma^2 - \alpha k^{2H}\right), \quad (32)$$

i.e. if $x > 233$, then $f(x,k)$ is replaced with its asymptotics for $x \to \infty$; see (32) and (23). Mathcad is able to calculate the first integral on the right-hand side of (32) in the range of parameters which is given in table 1.

The following results were obtained.

Table 1 – Verification of the obtained solution

$k=3$, $H=0.8$, $\alpha=\sigma=1$		
	$R(T+k)$	the integral (30)
$T=4$	–10.24934	–10.24935
$T=5$	–12.92881	–12.92884
$T=10$	–29.28861	–29.28809
$T=30$	–133.4598	–133.4534
$k=3$, $H=0.8$, $\alpha=\pi/2$, $\sigma=0.8$		
	$R(T+k)$	the integral (30)
$T=4$	–17.03041	–17.03042
$T=10$	–46.93724	–46.93725
$T=15$	–79.43844	–79.44034
$T=20$	–117.89713	–117.89911
$k=3$, $H=0.7$, $\alpha=\pi/2$, $\sigma=1.2$		
	$R(T+k)$	the integral (30)
$T=4$	–10.53367	–10.53367
$T=10$	–27.04465	–27.04475
$T=20$	–61.87533	–61.87372
$T=30$	–103.51651	–103.51531
$T=40$	–150.5975	–150.61689

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As can be seen from the table 1, $R(T+k)$ is in good agreement with the integral (30), so the solution (29) is true. In our opinion, the slight difference of the second and the third columns in table 1 is due to machine errors.

Of course, Mathcad could not adequately calculate the integral (30) at large values of T, i.e. at $T=10^3$, 10^4, etc. In order to verify the solution (29) for large values of T, we seek the asymptotics of the integrals in (30) if $T\to\infty$.

Let us denote

$$A=\frac{\Gamma(2H+1)-e^{p_0k}\Gamma(2H+1,\,p_0k)}{2H\Gamma(2H+1)}p_0\,, \tag{33}$$

then

$$\int_0^T d\tau J(\tau,k)R(T-\tau)=A\int_0^T dte^{p_0t}\left(\sigma^2-\frac{\alpha}{2}(T-t)^{2H}\right). \tag{34}$$

As is known [11],

$$\int_0^u x^{\nu-1}(u-x)^{\mu-1}e^{\beta x}dx= \\ =B(\mu,\nu)u^{\nu+\mu-1}\cdot {}_1F_1(\nu,\mu+\nu,\beta u)\,. \tag{35}$$

On the basis of (35) and (34) we have

$$\int_0^T d\tau J(\tau,k)R(T-\tau)=A\left(\frac{\sigma^2}{p_0}\left(e^{p_0T}-1\right)-\right. \\ \left.-\frac{\alpha}{2}T^{2H+1}\frac{\Gamma(2H+1)}{\Gamma(2H+2)}\cdot {}_1F_1(1,2H+2,p_0T)\right). \tag{36}$$

As is known [13], the function ${}_1F_1(\alpha,\beta,z)$ has the following asymptotics:

$${}_1F_1(\alpha,\beta,z)\Big|_{|z|\to\infty,-\frac{\pi}{2}<\arg z<\frac{3\pi}{2}}\sim \\ \sim\frac{\Gamma(\beta)e^z z^{\alpha-\beta}}{\Gamma(\alpha)}\sum_{s=0}^{\infty}\frac{(1-\alpha)_s(\beta-\alpha)_s}{s!}z^{-s}+ \\ +\frac{z^{-\alpha}e^{i\pi\alpha}}{\Gamma(\beta-\alpha)}\Gamma(\beta)\sum_{s=0}^{\infty}\frac{\alpha_s(\alpha-\beta+1)_s}{s!}(-z)^{-s}, \\ a_s\equiv a(a+1)(a+2)\ldots(a+s-1),\ a_0\equiv 1\,. \tag{37}$$

On the basis on (37) it can be seen that

$${}_1F_1(1,\beta,z)\Big|_{|z|\to\infty,-\frac{\pi}{2}<\arg z<\frac{3\pi}{2}}= \\ =\Gamma(\beta)e^z z^{1-\beta}-\frac{\Gamma(\beta)}{z\Gamma(\beta-1)}+O\left(\frac{1}{z^2}\right). \tag{38}$$

On the basis of (38), (36) and (25) we obtain

$$\int_0^T dtR(T-t)J(t,k)\Bigg|_{T\to\infty}\sim\frac{\alpha}{2}\frac{A}{p_0}T^{2H}\,. \tag{39}$$

It should be noticed that for $T\to\infty$ the integral in (39) behaves as a power-law rather than an exponent function! The integral of $I(t,k)$ is as follows:

$$\int_0^T dtR(T-t)I(t,k)=I_1-I_2\,, \\ I_1=\sigma^2\int_0^T dtI(t,k),\ I_2=\frac{\alpha}{2}\int_0^T dtI(t,k)(T-t)^{2H}\,. \tag{40}$$

After substituting $\xi=T-t$ into I_2 we obtain

$$I_2=\frac{\alpha}{2}\frac{1}{\pi}\int_0^{\infty}dxf(x,k)e^{-Tx}\int_0^T d\xi e^{\xi x}\xi^{2H}\,. \tag{41}$$

With the help of the tabulated integral

$$\int_0^u dxx^{\nu-1}e^{-\mu x}=\mu^{-\nu}\gamma(\nu,\mu u)\,, \\ \gamma(\alpha,x)\equiv\Gamma(\alpha)-\Gamma(\alpha,x) \tag{42}$$

and eqs. (41) and (20) we obtain

$$I_2=\frac{\alpha}{2}\frac{1}{\pi}T^{2H}\int_0^{\infty}dx\frac{f(x,k)}{x}g(x,T)\,, \\ g(x,T)=e^{-Tx}\sum_{n=0}^{\infty}\frac{(xT)^{n+1}}{n!(2H+1+n)}\,. \tag{43}$$

It should be noticed that

$$g(x,T)=e^{-Tx}\sum_{n=0}^{\infty}\frac{(xT)^{n+1}}{n!(2H+1+n)}\le \\ \le e^{-Tx}\sum_{n=0}^{\infty}\frac{(xT)^{n+1}}{(n+1)!}=e^{-Tx}\left(e^{Tx}-1\right)=1-e^{-Tx}\le 1\,, \tag{44}$$

so with account for (23) the integral in (43) is convergent; and the asymptotics of (43) for $T\to\infty$ is not larger than aT^{2H} where a is a constant. We assume that $I_2\big|_{T\to\infty}\sim T^{2H}$, this assumption is confirmed numerically in what follows.

As for I_1, we have

$$I_1=\sigma^2\int_0^T dtI(t,k)= \\ =\sigma^2\int_0^{\infty}dxf(x,k)\frac{1-e^{-xT}}{x}\ ;\ 1-e^{-xT}\le 1\,, \tag{45}$$

so obviously I_1 is bounded by a constant and $I_1=o(I_2)$ if $T\to\infty$.

Obviously

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$$\left(1-\frac{\alpha k^{2H}}{2\sigma^2}\right)R(T)\bigg|_{T\to\infty} \sim -\frac{\alpha}{2}\left(1-\frac{\alpha k^{2H}}{2\sigma^2}\right)T^{2H}, \quad R(T+k)\big|_{T\to\infty} \sim -\frac{\alpha}{2}T^{2H}. \tag{46}$$

So, the asymptotic behavior of the left-hand and right-hand sides of (1) on the basis of (29) for $T\to\infty$ are

$$R(T+k)\big|_{T\to\infty} \sim -\frac{\alpha}{2}T^{2H},$$
$$\int_0^T d\tau h(\tau,k)R(T-\tau)\bigg|_{T\to\infty} \sim \frac{\alpha}{2}T^{2H}\left(\frac{A}{p_0}-\right.$$
$$\left.-\left(1-\frac{\alpha k^{2H}}{2\sigma^2}\right)-\lim_{T\to\infty}\frac{1}{\pi}\int_0^\infty dx\frac{f(x,k)}{x}g(x,T)\right); \tag{47}$$

see also (39), (43) and (46). So if

$$\frac{A}{p_0}-\left(1-\frac{\alpha k^{2H}}{2\sigma^2}\right)-\lim_{T\to\infty}\frac{1}{\pi}\int_0^\infty dx\frac{f(x,k)}{x}g(x,T)= =-1, \tag{48}$$

then our solution (29) is true for $T\to\infty$.

The validity of (48) is checked numerically with the help of the Wolfram Mathematica 11 package, which is able to calculate the integral on the left-hand side of (48). The following results are obtained.

Table 2 – Verification of the obtained assymptotics

T	$\frac{A}{p_0}-\left(1-\frac{\alpha k^{2H}}{2\sigma^2}\right)-\frac{1}{\pi}\int_0^\infty dx\frac{f(x,k)}{x}g(x,T)$
$k=3$, $H=0.7$, $\alpha=\pi/2$, $\sigma=1.2$	
10^3	–0.999798
10^4	–0.9998
10^5	–0.999998
$k=4$, $H=0.8$, $\alpha=\pi/2$, $\sigma=0.8$	
T	$\frac{A}{p_0}-\left(1-\frac{\alpha k^{2H}}{2\sigma^2}\right)-\frac{1}{\pi}\int_0^\infty dx\frac{f(x,k)}{x}g(x,T)$
10^3	–0.996715
10^4	–0.999671
10^5	–0.999967

As can be seen from table 2, eq. (48) is valid, which justifies our solution.

5 RESULTS

An exact analytical solution to the Volterra integral equation (1) is obtained, see (29). The kernel of the corresponding Volterra integral equation is the correlation function of the continuous fractal process with the power-law structure function (3). Only the cases where the Hurst exponent $H\in(0.5;1)$ are considered. The obtained solution (29) is verified numerically. The asymptotic behavior of both sides of (1) for $T\to\infty$ is investigated. It is shown that our solution gives the correct asymptotic behavior.

6 DISCUSSION

The corresponding Volterra integral equation was discussed in [8] in the framework of the Kolmogorov–Wiener filter for the rather popular model with the structure function (3). It seems that the use of the Volterra integral equation in the framework of the Kolmogorov-Wiener filter is in some sense inadequate because the Fredholm integral equation of the first kind should be solved in order to obtain the Kolmogorov-Wiener filter output.

Nevertheless, the Volterra integral equation is of mathematical interest. It should also be noted that the Volterra integral equation is a special case of the Fredholm integral equation, which is rather popular in investigations of fractal processes, so the Volterra integral equation may be applied to some investigations of fractal processes.

An exact analytical solution to eq. (1) is obtained. It is shown that the term $I(t,k)$ in (29) which comes from the integrals over the banks of cut is convergent for any $t\geq 0$ if the Hurst exponent $H\in(0.5;1)$. In contrast to [8], $I(t,k)$ is a real-valued function. Also in contrast to [8], the residue part of (29) is taken into account. The obtained solution (29) is verified numerically on the basis of Mathcad 14 package.

The asymptotic behavior of both sides of (1) for $T\to\infty$ is also investigated on the basis of (29). It is shown that our solution gives the correct asymptotic behavior; the corresponding integral in (48) was taken numerically with the help of Wolfram Mathematica 11 package.

CONCLUSIONS

The Volterra integral equation (1) of the first kind the kernel of which is the correlation function (4) is solved.

The scientific novelty of the obtained results is that an exact analytic solution to the corresponding integral equation is obtained. The solution is verified numerically, it is also shown that our solution the gives correct asymptotic behavior. The results of the previous papers devoted to the integral equation under consideration are refined.

The practical significance is that the obtained results may be applied to investigations of fractal random processes.

Prospects for further research are to apply the obtained results to practical problems.

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ДО АНАЛІТИЧНОГО РОЗВ'ЯЗКУ ІНТЕГРАЛЬНОГО РІВНЯННЯ ВОЛЬТЕРИ ДЛЯ ДОСЛІДЖЕННЯ ФРАКТАЛЬНИХ ПРОЦЕСІВ

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АНОТАЦІЯ

Актуальність. Розглянуто інтегральне рівняння Вольтери першого роду, яке може бути застосовним до фільтрації та прогнозування випадкових фрактальних процесів, наприклад, у інформаційно-телекомунікаційних мережах та при керуванні складними технологічними процесами.

Метою роботи є отримати точний аналітичний розв'язок інтегрального рівняння Вольтери першого роду. Ядром відповідного інтегрального рівняння є кореляційна функція фрактального випадкового процесу, структурна функція якого є степеневою.

Метод. Інтегральне рівняння Вольтери першого роду розв'язано за допомогою стандартного методу перетворення Лапласа. Зворотне перетворення Лапласа приводить до контурного інтегралу від функції комплексної змінної. Цей інтеграл обчислено як суму частини, що містить лишок, та інтегралів вздовж берегів розрізу. Відповідні інтеграли пораховано за допомогою відомих розвинень спеціальних функцій.

Результати. Нами отримано точний аналітичний розв'язок інтегрального рівняння Вольтери, ядром якого є кореляційна функція фрактального випадкового процесу. Робота базується на моделі, в якій структурна функція відповідного фрактального процесу є степеневою функцією. Показано, що та частина розв'язку, яка не містить дельта-функції, є збіжною в будь-якій точці, якщо показник Херста є більшим за 0,5, тобто якщо процес має фрактальні властивості. Показано, що отриманий розв'язок є дійсною функцією. Отриманий розв'язок перевірено чисельно; також показано, що наш розв'язок дає правильну асимптотичну поведінку. Хоча отриманий розв'язок містить експоненційно зростаючу функцію часу, при великих часах інтеграл від отриманого розв'язку асимптотично веде себе як степенева функція.

Висновки. Важливо підкреслити, що нами отримано точний, а не наближений розв'язок інтегрального рівняння Вольтери, яке досліджується. Отриманий розв'язок може бути застосовним до фільтрації та прогнозування даних випадкового фрактального процесу. Як відомо, фрактальні процеси мають місце у величезній кількості різноманітних систем, тому результати цієї статті можуть мати широку область застосувань.

КЛЮЧОВІ СЛОВА: інтегральне рівняння Вольтери першого роду, показник Херста, перетворення Лапласа, фрактальний процес, точний аналітичний розв'язок.

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К АНАЛИТИЧЕСКОМУ РЕШЕНИЮ ИНТЕГРАЛЬНОГО УРАВНЕНИЯ ВОЛЬТЕРРЫ ДЛЯ ИССЛЕДОВАНИЯ ФРАКТАЛЬНЫХ ПРОЦЕССОВ

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АННОТАЦИЯ

Актуальность. Рассмотрено интегральное уравнение Вольтерры первого рода, которое может быть применено к фильтрации и прогнозированию случайных фрактальных процессов, например, в информационно-телекоммуникационных сетях и при управлении сложными технологическими процессами.

Целью работы является точное аналитическое решение интегрального уравнения Вольтерры первого рода. Ядром соответствующего интегрального уравнения является корреляционная функция фрактального случайного процесса, структурная функция которого является степенной.

Метод. Интегральное уравнение Вольтерры первого рода решено с помощью стандартного метода преобразования Лапласа. Обратное преобразование Лапласа приводит к контурному интегралу от функции комплексной переменной. Этот интеграл посчитан как сумма части, содержащей вычет, и интегралов вдоль берегов разреза. Соответствующие интегралы получены на основе известных разложений специальных функций.

Результаты. Нами получено точное аналитическое решение интегрального уравнения Вольтерры, ядром которого есть корреляционная функция фрактального случайного процесса. Работа основывается на модели, в которой структурная функция соответствующего фрактального процесса является степенной функцией. Показано, что та часть решения, которая не содержит дельта-функции, сходится в любой точке, если показатель Херста больше 0,5, то есть если процесс имеет фрактальные свойства. Показано, что полученное решение является действительной функцией. Полученное решение проверено численно; также показано, что наше решение дает правильное асимптотическое поведение. Хотя полученное решение содержит экспоненциально возрастающую функцию времени, при больших временах интеграл от полученного решения асимптотически ведет себя как степенная функция.

Выводы. Следует подчеркнуть, что нами получено точное, а не приближенное решение исследуемого интегрального уравнения Вольтерры. Полученное аналитическое решение может быть применено к фильтрации и прогнозированию данных случайного фрактального процесса. Как известно, фрактальные процессы имеют место в огромном количестве разных систем, поэтому результаты этой статьи могут иметь широкую область применения.

КЛЮЧЕВЫЕ СЛОВА: интегральное уравнение Вольтерры первого рода, показатель Херста, преобразование Лапласа, фрактальный процесс, точное аналитическое решение.

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