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QUEUEING SYSTEMS WITH DELAY

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ABSTRACT

Context. In the queuing theory of a research of the G/G/1 systems are relevant because it is impossible to receive decisions for the average waiting time in queue in a final form in case of arbitrary laws of distributions of an input flow and service time. Therefore, the study of such systems for particular cases of input distributions is important. The problem of deriving solutions for the average waiting time in a queue in closed form for systems with distributions shifted to the right from the zero point is considered.

Objective. Getting solutions for the main characteristics of the systems – the average waiting time of requirements in the queue for queuing systems (QS) of type G/G/1 with shifted input distributions.

Methods. To solve this problem, we used the classical method of spectral decomposition of the solution of the Lindley integral equation. This method allows to obtaining a solution for the average waiting time for two systems under consideration in a closed form. The method of spectral decomposition of the solution of the Lindley integral equation plays an important role in the theory of systems G/G/1. For the practical application of the results obtained, the well-known method of moments of probability theory is used.

Results. For the first time, spectral expansions are obtained for the solution of the Lindley integral equation for systems with delay, which are used to derive formulas for the average waiting time in a queue in closed form.

Conclusions. It is shown that in systems with delay, the average waiting time is less than in in the usual systems. The obtained formula for the average waiting time expands and complements the well-known queuing theory incomplete formula for the average waiting time for G/G/1 systems. This approach allows us to calculate the average latency for these systems in mathematical packages for a wide range of traffic parameters. In addition to the average waiting time, such an approach makes it possible to determine also moments of higher orders of waiting time. Given the fact that the packet delay variation (jitter) in telecommunications is defined as the spread of the waiting time from its average value, the jitter can be determined through the variance of the waiting time.

KEYWORDS: delayed system, shifted distributions, Laplace transform, Lindley integral equation, spectral decomposition method.

ABBREVIATIONS

LIE is a Lindley integral equation;

QS is a queuing system;

PDF is a probability distribution function.

NOMENCLATURE

$a(t)$ is a density function of the distribution of time between arrivals;

$A^*(s)$ is a Laplace transform of the function $a(t)$;

$b(t)$ is a density function of the distribution of service time;

$B^*(s)$ is a Laplace transform of the function $b(t)$;

c_λ is a coefficient of variation of time between arrivals;

c_μ is a coefficient of variation of service time;

E_2 is a Erlang distribution of the second order;

E_2^- is a shifted Erlang distribution of the second order;

G is a arbitrary distribution law;

H_2 is a hyperexponential distribution of the second order;

H_2^- is a shifted hyperexponential distribution of the second order;

HE_2 is a hypererlangian distribution of the second order;

HE_2^- is a shifted hypererlangian distribution of the second order;

M is a exponential distribution law;

M^- is a shifted exponential distribution law;

$\bar{W}$ is a average waiting time in the queue;

$W^*(s)$ is a Laplace transform of waiting time density function;

λ is a Erlang distribution parameter for input flow;

λ_1, λ_2 is a parameters of the hyperexponential (hyperelangian) distribution law of the input flow;

μ is a Erlang distribution parameter for of service time;

μ_1, μ_2 is a parameters of the hyperexponential (hyperelangian) distribution law of service time;

ρ is a system load factor;

$\bar{\tau}_\lambda$ is an average time between arrivals;

$\overline{\tau_\lambda^2}$ is a second initial moment of time between arrivals;

$\bar{\tau}_\mu$ is an average service time;

$\overline{\tau_\mu^2}$ is a second initial moment of service time;

$\Phi_+(s)$ is a Laplace transform of the PDF of waiting time;

$\psi_+(s)$ is a first component of spectral decomposition;

$\psi_-(s)$ is a second component of spectral decomposition.

INTRODUCTION

In the study of G/G/1 systems, an important role is played by the method of spectral decomposition of the solution of the Lindley integral equation (LIE) and most

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of the results in the queueing theory are obtained using this method. The most accessible this method with specific examples is described in the classic queueing theory [1].

This article is devoted to the analysis of QS with delay, i.e. systems defined by a pair of input distributions shifted to the right of the zero point. In [2] for the first time the results on the study of the M/M/1 system with delay with exponential input distributions ($M^- / M^- / 1$) shifted to the left from the zero point, obtained by the classical method of spectral decomposition, are presented. Hereinafter, the superscript "–" will denote the operation of shifting the distribution law.

In [2], it is shown that the average waiting time of a queue in the $M^- / M^- / 1$ system with delay is less than in the classical M/M/1 system with the same load factor due to the fact that the coefficients of variation of the arrival and service times become less than one with the lag parameter $t_0 > 0$. Thus, the operation of the shift of the distribution law transforms the classical Markov system M/M/1 into a non-Markov system $M^- / M^- / 1$.

The results of [2], together with [1], made it possible to extend the method of spectral decomposition of the solution of the Lindley integral equation into systems $H_2^- / H_2^- / 1$, $H_2^- / M^- / 1$ and $M^- / H_2^- / 1$ with delay [3]. In [4], results are given for a $HE_2^- / HE_2^- / 1$ system with shifted hypererlangian distributions, and in [5] – for systems $E_2^- / E_2^- / 1$, $E_2^- / M^- / 1$ and $M^- / E_2^- / 1$.

All the QS considered in the article, formed of the four most known shifted laws of distributions M^-, E_2^-, H_2^-, HE_2^- are of type G/G/1.

In the queueing theory, the studies of G/G/ systems are relevant due to the fact that they are actively used in modern teletraffic theory, moreover, it is impossible to obtain solutions for such systems in the final form for the general case.

The object of study is the queueing systems type G/G/1.

The subject of study is the main characteristics of the systems – the average waiting time of requirements in the queue.

The purpose of the work is obtaining a solution for the average waiting time of requirements in the queue in closed form for the above-mentioned systems.

1 PROBLEM STATEMENT

The paper poses the problem of finding the solution of the average waiting time of claims in a queue in the queueing systems, formed by four distribution laws shifted to the right from the zero point: M^-, E_2^-, H_2^-, HE_2^-. These four laws of distributions form 4x4=16 different QS G/G/1.

When using the method of spectral decomposition of an LIE solution to determine the average waiting time, we will follow the approach and symbolism of the author of the classical queuing theory [1]. To solve the problem, it is necessary to find the law of waiting time distribution in the system through the spectral decomposition of the form: $A^*(-s) \cdot B^*(s) - 1 = \psi_+(s) / \psi_-(s)$, where $\psi_+(s)$ and $\psi_-(s)$ are some fractional rational functions of s that can be factorized. Functions $\psi_+(s)$ and $\psi_-(s)$ must satisfy the following conditions according to [2]:

1) for $\mathrm{Re}(s) > 0$ function $\psi_+(s)$ is analytic without zeros in this half-plane;
2) for $\mathrm{Re}(s) < D$ function $\psi_-(s)$ is analytic without zeros in this half-plane, where D is some positive constant defined by the condition: $\lim_{t \to \infty} a(t) / e^{-Dt} < \infty$. (1)

In addition, functions $\psi_+(s)$ and $\psi_-(s)$ must have the following properties:

$$\lim_{|s| \to \infty, \mathrm{Re}(s) > 0} \frac{\psi_+(s)}{s} = 1, \quad \lim_{|s| \to \infty, \mathrm{Re}(s) < D} \frac{\psi_-(s)}{s} = -1. \quad (2)$$

To solve the problem, it is necessary first to construct for these systems spectral decompositions of the form $A^*(-s) \cdot B^*(s) - 1 = \psi_+(s) / \psi_-(s)$, taking into account conditions (1), (2) in each case.

2 REVIEW OF THE LITERATURE

The method of spectral decomposition of the solution of the Lindley integral equation was first presented in detail in the classic queueing theory [1], and was subsequently used in many papers, including [6, 7]. A different approach to solving Lindley's equation has been used in [8]. That work used factorization instead of the term "spectral decomposition" and instead of the functions $\psi_+(s)$ and $\psi_-(s)$ it used factorization components $\omega_+(z,t)$ and $\omega_-(z,t)$ of the function $1 - z \cdot \chi(t)$, where $\chi(t)$ is the characteristic function of a random variable ξ with an arbitrary distribution function $C(t)$, and z is any number from the interval (−1, 1). This approach for obtaining end results for systems under consideration is less convenient than the approach described and illustrated with numerous examples in [1].

The practical application of the method of spectral decomposition of an LIE solution for the study of systems with hyperexponential and exponential input distributions shown in [10].

In [2], for the first time, the results of an analysis of QS $M^- / M^- / 1$ with shifted exponential distributions are presented. In [3], results are given on systems $H_2^- / H_2^- / 1$, $H_2^- / M^- / 1$ and $M^- / H_2^- / 1$ with delay, in [4] – on the

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system $HE_2^- / HE_2^- / 1$ with delay, and in [5] – on the systems $E_2^- / E_2^- / 1$, $E_2^- / M^- / 1$ and $M^- / E_2^- / 1$.

In [9] presents the results of the approach of queues to the Internet and mobile services as queues with a delay in time. It is shown that if information is delayed long enough, a Hopf bifurcation can occur, which can cause unwanted fluctuations in the queues. However, it is not known how large the fluctuations are when the Hopf bifurcation occurs. This is the first publication in the English-language journals about queues with a delay. Approximate methods with respect to the laws of distributions are described in detail in [13–15], and similar studies in queuing theory have recently been carried out in [16–18].

3 MATERIALS AND METHODS

Consider the class of density functions $f(t)$, which are Laplace-convertible, that is, for which there is a function $F^*(s) = \int_0^\infty e^{-st} f(t)dt \equiv L[f(t)]$. Next, we use the delay theorem as a property of the Laplace transform: for any $t_0 > 0$, the equality will be satisfied

$$L[f(t-t_0)] = e^{-st_0} \cdot F^*(s), \qquad (3)$$

where $\mathrm{Re}(s) > 0$. The considered density functions M, E_2, H_2, HE_2 belong to this class.

In [2–5], using equality (3), obtained spectral decompositions $A^*(-s)\cdot B^*(s) - 1 = \psi_+(s)/\psi_-(s)$ and derived formulas for the average waiting time for eight systems. Based on these results, we can now formulate a general statement.

Statement. Spectral expansions $A^*(-s) * B^*(s) - 1 = \psi_+(s)/\psi_-(s)$ of the LIE solution for all sixteen systems under consideration with delay completely coincide with spectral expansions for the corresponding ordinary systems, i.e. the main expression $A^*(-s) * B^*(s) - 1$ of the spectral decomposition is invariant to the operation of the time shift of the density function.

The proof is carried out on the example of the $M^- / M^- / 1$ system.

The density function of the distribution of intervals between the requirements of the input flow in this case is equal to

$$a(t) = \begin{cases} \lambda e^{-\lambda(t-t_0)}, & t \ge t_0 \\ 0, & 0 \le t < t_0 \end{cases}, \qquad (4)$$

and for service time

$$b(t) = \begin{cases} \mu e^{-\mu(t-t_0)}, & t \ge t_0 \\ 0, & 0 \le t < t_0 \end{cases}. \qquad (5)$$

Functions (4) and (5) are exponential distributions shifted to the right of the zero point.

Let us write the Laplace transforms of functions (4) and (5) taking into account property (3):

$$A^*(s) = \frac{\lambda e^{-t_0 s}}{s+\lambda}, \quad B^*(s) = \frac{\mu e^{-t_0 s}}{s+\mu}.$$

Then the expression for the spectral decomposition of the solution of the LIE for the $M^- / M^- / 1$ system will take the form:

$$\frac{\psi_+(s)}{\psi_-(s)} = \frac{\lambda e^{t_0 s}}{\lambda - s} \cdot \frac{\mu e^{-t_0 s}}{\mu + s} - 1 = \frac{\lambda\mu - (\lambda - s)(\mu + s)}{(\lambda - s)(\mu + s)} = \frac{s(s+\mu-\lambda)}{(\lambda - s)(\mu + s)}. \qquad (6)$$

Here, exponents with opposite signs of exponential functions are reset to zero, and thus the shift operation in the spectral decomposition is leveled. The right-hand side of expression (6) completely coincides with the spectral decomposition of the solution of an LIE for the classical system M/M/1. It will be similar for other systems, the spectral expansions for systems with delay and ordinary systems will coincide [2–5].

Assertion is proved.

Corollary. The formulas for the average waiting time for all systems with shifted distributions will have exactly the same form as for the corresponding systems with ordinary distributions, but with changed parameters due to the time shift operation [2–5]. Consequently, the average waiting time for systems with lag actually depends on the magnitude of the shift parameter $t_0 > 0$.

Further, taking into account conditions (1) and (2), we construct functions $\psi_+(s)$ and $\psi_-(s)$, and using the latter, we determine the Laplace transform of the waiting time density function. So, for the $M^- / M^- / 1$ system we have:

$$\psi_+(s) = \frac{s(s+\mu-\lambda)}{(\mu+s)}, \quad \psi_-(s) = \lambda - s.$$

Note that the function $\psi_+(s)$ has no zeros and poles in the half-plane $\mathrm{Re}(s) > 0$, and the function $\psi_-(s)$ has no zeros in the half-plane $\mathrm{Re}(s) < \lambda$.

According to the method of spectral decomposition, the constant K is determined from the condition: $K = \lim_{s\to 0} \frac{\psi_+(s)}{s} = \lim_{s\to 0} \frac{s+\mu-\lambda}{s+\mu} = 1 - \lambda/\mu$, where the parameters λ and μ are determined by expressions (7) and (8) by the method of moments:

$$\bar{\tau}_\lambda = \lambda^{-1} + t_0, \qquad (7)$$

$$\bar{\tau}_\mu = \mu^{-1} + t_0. \qquad (8)$$

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The constant K determines the probability that the demand entering the system finds it free.

In this case, the ratio λ/μ does not determine the load factor as in the case of the M/M/1 system and the parameters λ and μ are not the intensity of input flow and service time, respectively. The Laplace transform for the waiting time distribution function, following [1], has the form: $\Phi_+(s) = \frac{K}{\psi_+(s)} = \frac{(1-\lambda/\mu)(\mu+s)}{(s+\mu-\lambda)}$, and the Laplace transform of the waiting time density function

$$W^*(s) = s\Phi_+(s) = \frac{(1-\lambda/\mu)(\mu+s)}{(s+\mu-\lambda)}. \qquad (9)$$

Then the average waiting time in the queue for the QS $M^-/M^-/1$ is equal to the value of the derivative of the Laplace transform (9) of the density function with a minus sign at the point $s=0$:

$-\frac{dW^*(s)}{ds}|s=0 = \frac{\lambda/\mu}{\mu-\lambda}$. Finally, the average waiting time in the queue for QS $M^-/M^-/1$

$$\overline{W} = \frac{\lambda/\mu}{\mu-\lambda}. \qquad (10)$$

Distributions (4) and (5) contain two parameters; to determine them by the moments method, the moment equations (7) and (8) must be supplemented with expressions for the second-order initial moments. As the second moments it is more convenient to use the coefficients of variation:

$$c_\lambda = (1+\lambda t_0)^{-1}, \qquad (11)$$

$$c_\mu = (1+\mu t_0)^{-1}. \qquad (12)$$

Then, as input parameters for calculating the QS $M^-/M^-/1$, we set the values $t_0, \overline{\tau}_\lambda, c_\lambda, \overline{\tau}_\mu, c_\mu$, and the unknown parameters λ, μ and t_0 are determined from the system of moment equations (7), (8), (11) and (12).

This system of equations is overdetermined and the input parameters will be bound by the condition

$$c_\mu = 1-(1-c_\lambda)/\rho, \qquad (13)$$

where $\rho = \overline{\tau}_\mu / \overline{\tau}_\lambda$ is load factor.

Similar reasoning for other systems with a delay will lead to similar results. To do this, in Table 1 we give the numerical characteristics of the considered laws of distributions, which were used in [2–5].

The numerical characteristics of the shifted distributions (Table 1) clearly indicate a significant influence on them of the shift parameter t0. The numerical characteristics of the shifted distributions (Table 1) clearly indicate a significant influence on them of the shift parameter t_0. Now it is necessary to determine the unknown parameters of these distributions. These parameters were also obtained in [2–5] and for the cases of density functions of the distribution of intervals of input flows $a(t)$ are given in Table 2. Similar parameters for the service time distributions $b(t)$ will take place by replacing λ with μ.

Table 3 shows the Laplace transformations of the waiting time density functionы in the queues in the systems under consideration, the components of the spectral expansions of the LIE solution, as well as the expressions for the average waiting time in the corresponding systems.

A detailed description of the algorithms for calculating the average waiting time for the systems under consideration can be found in [2–5]. In this way, are the published results for eight of the sixteen systems.

Table 1 – Numerical Characteristics of Distributions

Distribution laws	$\overline{\tau}_\lambda$	$\overline{\tau^2_\lambda}$	c^2_λ
M	$1/\lambda$	$2/\lambda^2$	1
E_2	$1/\lambda$	$3/(2\lambda^2)$	1/2
H_2	$p/\lambda_1+(1-p)/\lambda_2$	$2[p/\lambda_1^2+(1-p)/\lambda_2^2]$	$\frac{(1-p^2)\lambda_1^2-2\lambda_1\lambda_2 p(1-p)+p(2-p)\lambda_2^2}{[(1-p)\lambda_1+p\lambda_2]^2}$
HE_2	$p/\lambda_1+(1-p)/\lambda_2$	$3p/(2\lambda_1^2)+3(1-p)/(2\lambda_2^2)$	$\frac{\lambda_1^2-2p\lambda_2(\lambda_1-\lambda_2)+p(1-2p)(\lambda_1-\lambda_2)^2}{2[(1-p)\lambda_1+p\lambda_2]^2}$
M^-	$\frac{1}{\lambda}+t_0$	$2(\frac{1}{\lambda^2}+\frac{t_0}{\lambda})+t_0^2$	$\frac{1}{(1+\lambda t_0)^2}$
E_2^-	$\frac{1}{\lambda}+t_0$	$\frac{3}{2\lambda^2}+2\frac{t_0}{\lambda}+t_0^2$	$\frac{1}{2(1+\lambda t_0)^2}$
H_2^-	$\frac{p}{\lambda_1}+\frac{(1-p)}{\lambda_2}+t_0$	$t_0^2+2t_0[\frac{p}{\lambda_1}+\frac{(1-p)}{\lambda_2}]+2[\frac{p}{\lambda_1^2}+\frac{(1-p)}{\lambda_2^2}]$	$\frac{(1-p^2)\lambda_1^2-2\lambda_1\lambda_2 p(1-p)+p(2-p)\lambda_2^2}{[t_0\lambda_1\lambda_2+(1-p)\lambda_1+p\lambda_2]^2}$
HE_2^-	$\frac{p}{\lambda_1}+\frac{(1-p)}{\lambda_2}+t_0$	$t_0^2+2t_0[\frac{p}{\lambda_1}+\frac{(1-p)}{\lambda_2}]+\frac{3p}{2\lambda_1^2}+\frac{3(1-p)}{2\lambda_2^2}$	$\frac{\lambda_1^2-2p\lambda_2(\lambda_1-\lambda_2)+p(1-2p)(\lambda_1-\lambda_2)^2}{2[t_0\lambda_1\lambda_2-p(\lambda_1-\lambda_2)+\lambda_1]^2}$

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Table 2 – The parameters of the shifted distributions obtained by the method of moments

Distribution laws	Density function $a(t)$	Parameters $p, \lambda, \lambda_1, \lambda_2$		
M^-	$\lambda e^{-\lambda(t-t_0)}$	$\lambda = \frac{1}{\bar{\tau}_\lambda - t_0}$		
E_2^-	$4\lambda^2 (t-t_0) e^{-2\lambda(t-t_0)}$	$\lambda = \frac{1}{\bar{\tau}_\lambda - t_0}$		
H_2^-	$p\lambda_1 e^{-\lambda_1(t-t_0)} + (1-p)\lambda_2 e^{-\lambda_2(t-t_0)}$	$p = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{(\bar{\tau}_\lambda - t_0)^2}{2[(\bar{\tau}_\lambda - t_0)^2 + c_\lambda^2 \bar{\tau}_\lambda^2]}}$	$\lambda_1 = \frac{2p}{(\bar{\tau}_\lambda - t_0)}$	$\lambda_2 = \frac{2(1-p)}{(\bar{\tau}_\lambda - t_0)}$
HE_2^-	$4p\lambda_1^2 (t-t_0) e^{-2\lambda_1(t-t_0)} + 4(1-p)\lambda_2^2 (t-t_0) e^{-2\lambda_2(t-t_0)}$	$p = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{3(\bar{\tau}_\lambda - t_0)^2}{8[(\bar{\tau}_\lambda - t_0)^2 + c_\lambda^2 \bar{\tau}_\lambda^2]}}$	$\lambda_1 = \frac{2p}{(\bar{\tau}_\lambda - t_0)}$	$\lambda_2 = \frac{2(1-p)}{(\bar{\tau}_\lambda - t_0)}$

Table 3 – The Laplace transform of the waiting time density function, the components of the spectral decompositions of the LIE solution, the expressions for the mean waiting time

QS	The Laplace transform of the waiting time density function and the components of the spectral decompositions	The expressions for the average waiting time
$M^- / M^- / 1$	$W^*(s) = \frac{(1-\lambda/\mu)(\mu+s)}{(s+\mu-\lambda)}$, $\psi_+(s) = \frac{s(s+\mu-\lambda)}{(\mu+s)}$, $\psi_-(s) = \lambda - s$.	$\bar{W} = \frac{\lambda/\mu}{\mu-\lambda}$.
$M^- / E_2^- / 1$	$W^*(s) = \frac{(1-\rho)(2\mu+s)^2}{(s+s_1)(s+s_2)}$, $\psi_+(s) = \frac{s(s+s_1)(s+s_2)}{(2\mu+s)^2}$, $\psi_-(s) = \lambda - s$.	$\bar{W} = \frac{3\rho}{4\mu(1-\rho)}$.
$M^- / H_2^- / 1$	$W^*(s) = \frac{s_1 s_2 (s+\mu_1)(s+\mu_2)}{\mu_1\mu_2 (s+s_1)(s+s_2)}$, $\psi_+(s) = \frac{s(s+s_1)(s+s_2)}{(s+\mu_1)(s+\mu_2)}$, $\psi_-(s) = \lambda - s$.	$\bar{W} = \frac{s_1+s_2}{s_1 s_2} - \frac{\mu_1+\mu_2}{\mu_1\mu_2}$, where s_1, s_2 absolute values of negative zeros of the function numerator $\psi_+(s)/\psi_-(s)$.
$E_2^- / E_2^- / 1$	$W^*(s) = \frac{s_1 s_2 (2\mu+s)^2}{4\mu^2 s(s+s_1)(s+s_2)}$, $\psi_+(s) = \frac{s(s+s_1)(s+s_2)}{(2\mu+s)^2}$, $\psi_-(s) = -\frac{(2\lambda-s)^2}{(s-s_3)}$.	$\bar{W} = \frac{s_1+s_2}{s_1 s_2} - \frac{1}{\mu}$, where $s_1 = (\mu-\lambda) + \sqrt{(\mu-\lambda)^2 + 8\lambda\mu}$, $s_2 = 2(\mu-\lambda)$.
$E_2^- / M^- / 1$	$W^*(s) = \frac{s_1(s+\mu)}{\mu(s+s_1)}$, $\psi_+(s) = \frac{s(s+s_1)}{s+\mu}$, $\psi_-(s) = -\frac{(2\lambda-s)^2}{s-s_2}$.	$\bar{W} = 1/s_1 - 1/\mu$, where $s_1 = (\mu-4\lambda)/2 + \sqrt{[(\mu-4\lambda)/2]^2 + 4\lambda(\mu-\lambda)}$.
$H_2^- / H_2^- / 1$	$W^*(s) = \frac{s_1 s_2 (s+\mu_1)(s+\mu_2)}{\mu_1\mu_2 (s+s_1)(s+s_2)}$, $\psi_+(s) = \frac{s(s+s_1)(s+s_2)}{(s+\mu_1)(s+\mu_2)}$, $\psi_-(s) = \frac{(s-\lambda_1)(\lambda_2-s)}{s-s_3}$.	$\bar{W} = \frac{1}{s_1} + \frac{1}{s_2} - \frac{1}{\mu_1} - \frac{1}{\mu_2}$, where s_1, s_2 absolute values of negative zeros of the function numerator $\psi_+(s)/\psi_-(s)$.
$H_2^- / M^- / 1$	$W^*(s) = \frac{s_1(s+\mu)}{\mu(s+s_1)}$, $\psi_+(s) = \frac{s(s+s_1)}{s+\mu}$, $\psi_-(s) = \frac{(s-\lambda_1)(\lambda_2-s)}{s-s_2}$.	$\bar{W} = 1/s_1 - 1/\mu$, where s_1 absolute values of negative zeros of the function numerator $\psi_+(s)/\psi_-(s)$.
$HE_2^- / HE_2^- / 1$	$W^*(s) = \frac{s_1 s_2 s_3 s_4 (s+2\mu_1)^2 (s+2\mu_2)^2}{16\mu_1^2\mu_2^2 (s+s_1)(s+s_2)(s+s_3)(s+s_4)}$, $\psi_+(s) = \frac{s(s+s_1)(s+s_2)(s+s_3)(s+s_4)}{[(s+2\mu_1)^2 (s+2\mu_2)^2]}$, $\psi_-(s) = -\frac{(2\lambda_1-s)^2 (2\lambda_2-s)^2}{(s-s_5)(s-s_6)(s-s_7)}$.	$\bar{W} = \frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} + \frac{1}{s_4} - \frac{1}{\mu_1} - \frac{1}{\mu_2}$, where s_1, s_2, s_3, s_4 absolute values of negative zeros of the function numerator $\psi_+(s)/\psi_-(s)$.

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4 EXPERIMENTS

Tables 4–7 below show the data of calculations in the Mathcad package for four characteristic systems $M^{-}/M^{-}/1$, $E_2^{-}/E_2^{-}/1$, $H_2^{-}/H_2^{-}/1$, $HE_2^{-}/HE_2^{-}/1$ for cases of low, medium and high load $\rho = 0{,}1; 0{,}5; 0{,}9$ for a wide range of variation coefficients of variation c_λ, c_μ and a shift parameter t_0. Results for systems with a delay are compared with results for usual systems. It is obvious that the average waiting time in a system with a delay depends on the shift parameter t_0.

The load factor ρ in both tables is determined by the ratio of average intervals $\rho = \overline{\tau}_\mu / \overline{\tau}_\lambda$. The calculations used the normalized service time $\overline{\tau}_\mu = 1$.

Table 4 – Results of experiments for QS $M^{-}/M^{-}/1$ and M/M/1

Input parameters				Average waiting time	
ρ	t_0	c_λ	c_μ	For QS $M^{-}/M^{-}/1$	For QS M/M/1
0.1	0.9	0.91	0.1	0.001	0.111
	0.5	0.95	0.5	0.028	
	0.1	0.99	0.9	0.090	
	0.01	0.999	0.99	0.109	
0.5	0.9	0.55	0.1	0.010	1.000
	0.5	0.75	0.5	0.250	
	0.1	0.95	0.9	0.810	
	0.01	0.995	0.99	0.980	
0.9	0.9	0.19	0.1	0.090	9.000
	0.5	0.55	0.5	2.250	
	0.1	0.91	0.9	7.290	
	0.01	0.991	0.99	8.821	

Table 5 – Results of experiments for QS $E_2^{-}/E_2^{-}/1$ and $E_2/E_2/1$

Input parameters				Average waiting time	
ρ	c_λ	c_μ	t_0	For QS $E_2^{-}/E_2^{-}/1$	For QS $E_2/E_2/1$
0.1	0.643	0.071	0.9	0.000	0.017
	0.672	0.354	0.5	0.002	
	0.700	0.636	0.1	0.013	
	0.706	0.700	0.01	0.016	
0.5	0.389	0.071	0.9	0.001	0.390
	0.530	0.354	0.5	0.081	
	0.672	0.636	0.1	0.309	
	0.704	0.700	0.01	0.382	
0.9	0.134	0.071	0.9	0.034	4.359
	0.389	0.354	0.5	1.057	
	0.643	0.636	0.1	3.519	
	0.701	0.700	0.01	4.271	

5 RESULTS

As one would expect, a decrease in the coefficients of variation c_λ and c_μ due to the introduction of the shift parameter $t_0 > 0$ into the laws of the distributions of the input flow and service time, entails a decrease in the average waiting time in systems with a delay several times.

Table 6 – Results of experiments for QS $H_2^{-}/H_2^{-}/1$ and $H_2/H_2/1$

Input parameters		Average waiting time			
ρ	(c_λ, c_μ)	For QS $H_2^{-}/H_2^{-}/1$			For QS $H_2/H_2/1$
		t_0=0.9	t_0=0.5	t_0=0.1	
0.1	(1.1)	0.06	0.07	0.10	0.11
	(2.2)	0.28	0.36	0.42	0.45
	(4.4)	1.19	1.54	1.73	1.78
	(8.8)	4.81	6.31	6.97	7.11
0.5	(1.1)	0.56	0.75	0.95	1.00
	(2.2)	2.31	3.13	3.87	4.04
	(4.4)	9.29	12.61	15.45	16.13
	(8.8)	37.22	50.50	61.54	64.18
0.9	(1.1)	6.04	8.30	8.91	9.00
	(2.2)	24.14	33.22	35.84	36.20
	(4.4)	96.51	132.30	143.27	144.83
	(8.8)	386.03	527.68	571.47	577.86

Table 7 – Results of experiments for QS $HE_2^{-}/HE_2^{-}/1$ and $HE_2/HE_2/1$

Input parameters		Average waiting time			
ρ	$(c_\lambda; c_\mu)$	For QS $HE_2^{-}/HE_2^{-}/1$			For QS $HE_2/HE_2/1$
		t_0=0.9	t_0=0.5	t_0=0.1	
0.1	(0.71;0.71)	0.01	0.01	0.02	0.02
	(2;2)	0.29	0.32	0.33	0.34
	(4;4)	1.21	1.55	1.66	1.68
	(8;8)	4.93	6.48	7.05	7.16
0.5	(0.71;0.71)	0.27	0.31	0.39	0.40
	(2;2)	2.32	3.15	3.82	3.98
	(4;4)	9.35	12.94	15.85	16.53
	(8;8)	37.50	52.13	63.96	66.73
0.9	(0.71;0.71)	3.06	4.11	4.39	4.40
	(2;2)	24.42	33.36	35.88	36.21
	(4;4)	97.71	133.3	143.8	145.31
	(8;8)	390.90	532.7	574.5	580.56

The adequacy of the presented results is fully confirmed by the fact that when the shift parameter t_0 tends to zero, the average waiting time in the delayed system tends to its value in the usual system. The above calculation results are in good agreement with the results of work [11] in the range of parameters in which the systems under consideration are valid.

6 DISCUSSION

The operation of the shift in time on the one hand, leads to an increase in system load with a delay. For example, for a $M^{-}/M^{-}/1$ system with a delay, the load increases by a factor of $(1+\mu t_0)/(1+\lambda t_0)$ compared to a usual M/M/1 system.

The time shift operation, on the other hand, reduces the variation coefficients of the interval between receipts and of the service time of requirements. Since the average waiting time in the system G/G/1 is related to the coefficients of variation of the arrival and servicing intervals by a quadratic dependence, the average waiting time in the delayed system will be less than in the usual system under the same load factor.

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For example, for the $M^{-}/M^{-}/1$ system when loading ρ=0,9 and the shift parameter t_0=0,9, the variation coefficient c_λ of the interval between receipts decreases from 1 for a usual system to 0,19, the service time variation coefficient c_μ decreases from 1 to 0,1, and the waiting time decreases from 9 units of time for a usual system to almost 0.,09 units of time for a delayed system (Table 4).

In addition, the introduction of the shift parameter leads to a fairly wide range of variation in the coefficients of variation c_λ and c_μ, in contrast to usual systems, which are applicable only in the case of fixed values of the coefficients of variation. Therefore, systems with delay extend the range of their applicability in the modern theory of teletraffic.

CONCLUSIONS

The paper presents the spectral expansions of the solution of the Lindley integral equation for eight systems with delay, which are used to derive expressions for the average waiting time in the queue for these systems in closed form.

The scientific novelty of the results is that spectral expansions of the solution of the Lindley integral equation for the systems under consideration are obtained and with their help the calculated expressions for the average waiting time in the queue for systems with delay in closed form are derived. These expressions complements and expands the well-known incomplete formula for the average waiting time in the G/G/1 systems with arbitrary laws of input flow distribution and service time.

The practical significance of the work lies in the fact that the obtained results can be successfully applied in the modern theory of teletraffic, where the delays of incoming traffic packets play a primary role. For this, it is necessary to know the numerical characteristics of the incoming traffic intervals and the service time at the level of the first two moments, which does not cause difficulties when using modern traffic analyzers [10].

Prospects for further research are seen in the continuation of the study of systems of type G/G/1 with other common input distributions and in expanding and supplementing the formulas for average waiting time.

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СИСТЕМИ МАСОВОГО ОБСЛУГОВУВАННЯ З ЗАПІЗНЕННЯМ

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АНОТАЦІЯ

Актуальність. У теорії масового обслуговування дослідження систем G/G/1 актуальні в зв'язку з тим, що не можна отримати рішення для часу очікування в кінцевому вигляді в загальному випадку при довільних законах розподілів вхідного потоку і часу обслуговування. Тому важливі дослідження таких систем для окремих випадків вхідних розподілів. Розглянуто задачу виведення рішень для середнього часу очікування в черзі в замкнутій формі для систем зі зсунутими вправо від нульової точки вхідними розподілами.

Мета роботи. Отримання рішення для основної характеристики системи – середнього часу очікування вимог в черзі для двох систем масового обслуговування типу G/G/1 зі зсунутими вхідними розподілами.

Метод. Для вирішення поставленого завдання був використаний класичний метод спектрального розкладання рішення інтегрального рівняння Ліндлі. Цей метод дозволяє отримати рішення для середнього часу очікування для розглянутих систем в замкнутій формі. Метод спектрального розкладання рішення інтегрального рівняння Ліндлі грає важливу роль в теорії систем G/G/1. Для практичного застосування отриманих результатів було використано відомий метод моментів теорії ймовірностей.

Результати. Вперше отримано спектральні розкладання рішення інтегрального рівняння Линдли для розглянутих систем, за допомогою яких виведені розрахункові вирази для середнього часу очікування в черзі в замкнутій формі.

Висновки. Показано, що в системах з запізненням у часі середній час очікування менше, ніж в звичайних системах. Отримані розрахункові вирази для часу очікування розширюють і доповнюють відому незавершену формулу теорії масового обслуговування для середнього часу очікування для систем G/G/1. Такий підхід дозволяє розрахувати середній час очікування для зазначених систем в математичних пакетах для широкого діапазону зміни параметрів трафіку. Отримані результати з успіхом можуть бути застосовані в сучасній теорії телетрафіка, де затримки пакетів вхідного трафіку відіграють першорядну роль. Дані результати з успіхом можуть бути застосовані в сучасній теорії телетрафіка, де затримки пакетів вхідного трафіку відіграють першорядну роль. Крім середнього часу очікування, такий підхід дає можливість також визначити моменти вищих порядків часу очікування. З огляду на той факт, що варіація затримки пакетів (джиттер) в телекомунікації визначається як дисперсія часу очікування від його середнього значення, то джиттер можна буде визначити через дисперсію часу очікування.

КЛЮЧОВІ СЛОВА: система з запізненням, зсунуті розподілу, перетворення Лапласа, інтегральне рівняння Линдли, метод спектрального розкладання.

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СИСТЕМЫ МАССОВОГО ОБСЛУЖИВАНИЯ С ЗАПАЗДЫВАНИЕМ

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АННОТАЦИЯ

Актуальность. В теории массового обслуживания исследования систем G/G/1 актуальны в связи с тем, что нельзя получить решения для времени ожидания в конечном виде в общем случае при произвольных законах распределений входного потока и времени обслуживания. Поэтому важны исследования таких систем для частных случаев входных распределений. Рассмотрена задача вывода решений для среднего времени ожидания в очереди в замкнутой форме для систем со сдвинутыми вправо от нулевой точки входными распределениями.

Цель работы. Получение решения для основной характеристики систем – среднего времени ожидания требований в очереди для систем массового обслуживания (СМО) типа G/G/1 со сдвинутыми входными распределениями.

Метод. Для решения поставленной задачи использован классический метод спектрального разложения решения интегрального уравнения Линдли. Данный метод позволяет получить решение для среднего времени ожидания для рассматриваемых систем в замкнутой форме. Метод спектрального разложения решения интегрального уравнения Линдли играет важную роль в теории систем G/G/1. Для практического применения полученных результатов использован известный метод моментов теории вероятностей.

Результаты. Впервые получены спектральные разложения решения интегрального уравнения Линдли для систем, с помощью которых выведены расчетные выражения для среднего времени ожидания в очереди в замкнутой форме.

Выводы. Получены спектральные разложения решения интегрального уравнения Линдли для рассматриваемых систем и с их помощью выведены расчетные выражения для среднего времени ожидания в очереди для этих систем в замкнутой форме. Показано, что в системах с запаздыванием во времени среднее время ожидания меньше, чем в обычных системах. Полученные расчетные выражения для времени ожидания расширяют и дополняют известную незавершенную формулу теории массового обслуживания для среднего времени ожидания для систем G/G/1. Такой подход позволяет рассчитать среднее время ожидания для указанных систем в математических пакетах для широкого диапазона изменения параметров трафика. Кроме среднего времени ожидания, такой подход дает возможность определить и моменты высших порядков времени ожидания. Учитывая тот факт, что вариация задержки пакетов (джиттер) в телекоммуникациях определяется как разброс времени ожидания от его среднего значения, то джиттер можно будет определить через дисперсию времени ожидания.

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КЛЮЧЕВЫЕ СЛОВА: система с запаздыванием, сдвинутые распределения, преобразование Лапласа, интегральное уравнение Линдли, метод спектрального разложения.

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