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MODEL OF TELETRAFFIC BASED ON QUEUEING SYSTEMS $E_2/HE_2/1$ WITH ORDINARY AND SHIFTED INPUT DISTRIBUTIONS

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ABSTRACT

Context. The study of G/G/1 systems is related to their relevance in the modern theory of teletraffic and, therefore, in the theory of computing systems and networks. In turn, this follows from the fact that it is impossible to obtain solutions for the waiting time in these systems in the final form in the general case with arbitrary laws of the distribution of the input flow and service time. Therefore, the study of such systems for particular cases of input distributions is important.

Objective. Obtaining a solution for the main system characteristic – the average waiting time in queue for two queuing systems of type G/G/1 with conventional and with shifted second-order Erlang and Hyper-Erlang input distributions.

Method. To solve this problem, we used the classical spectral decomposition method for solving the Lindley integral equation, which plays an important role in the theory of G/G/1 systems. This method allows obtaining a solution for the average waiting time for the considered systems in a closed form. For the practical application of the obtained results, the well-known probability theory moments method is used.

Results. For the first time, spectral expansions of the solution of the Lindley integral equation are obtained for two systems, with the help of which the formulas for the average waiting time in the queue are derived in closed form. The system with shifted Erlang and Hyper-Erlang input distributions provides shorter waiting times for requirements in the queue compared to a conventional system by reducing the coefficients of variation of intervals between requirements and of service time.

Conclusions. Spectral expansions of the solution of the Lindley integral equation for the systems under consideration are obtained and their complete coincidence is proved. Consequently, the formulas for the average waiting time in the queue for these systems are the same, but with modified parameters. These formulas expand and supplement the known queuing theory incomplete formula for the average waiting time for G/G/1 systems with arbitrary laws distributions of input flow and service time. This approach allows us to calculate the average latency for these systems in mathematical packages for a wide range of traffic parameters. All other characteristics of the systems are derived from the waiting time. In addition to the average waiting time, such an approach makes it possible to determine also moments of higher orders of waiting time. Given the fact that the packet delay variation (jitter) in telecommunications is defined as the spread of the waiting time from its average value, the jitter can be determined through the variance of the waiting time. The results are published for the first time.

KEYWORDS: Erlang and Hyper-Erlang distribution laws, Lindley integral equation, spectral decomposition method, Laplace transform.

ABBREVIATIONS

LIE is a Lindley integral equation;
QS is a queuing system;
PDF is a probability distribution function.

NOMENCLATURE

$a(t)$ is a density function of the distribution of time between arrivals;

$A^*(s)$ is a Laplace transform of the function $a(t)$;

$b(t)$ is a density function of the distribution of service time;

$B^*(s)$ is a Laplace transform of the function $b(t)$;

c_λ the coefficient of variation of time between arrivals;

c_μ the coefficient of variation of service time;

E_2 is a erlangian distribution of the second order;

E_2^- is a shifted erlangian distribution of the second order;

HE_2 is a hypererlangian distribution of the second order;

HE_2^- is a shifted hypererlangian distribution of the second order;

G is a arbitrary distribution law;

M –экспоненциальный закон распределения;
is a exponential distribution law;

$\bar{W}$ is a average waiting time in the queue;

$W^*(s)$ is a Laplace transform of waiting time density function;

λ is a parameters of the erlangian distribution law of the input flow;

μ_1, μ_2 is a parameters of the hyperelangian distribution law of service time;

ρ is a system load factor;

$\bar{\tau}_\lambda$ the average time between arrivals;

$\overline{\tau_\lambda^2}$ the second initial moment of time between arrivals;

$\bar{\tau}_\mu$ the average service time;

$\overline{\tau_\mu^2}$ the second initial moment of service time;

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$\Phi_+(s)$ the Laplace transform of the PDF of waiting time;

$\psi_+(s)$ is a first component of spectral decomposition;

$\psi_-(s)$ is a second component of spectral decomposition.

INTRODUCTION

This article is devoted to the analysis of $E_2/HE_2/1$ QS with ordinary and with shifted Erlang (E_2) and Hyper-Erlang (HE_2) input distributions and is a continuation of the work of the authors on systems with a delay. In the public domain, the authors were unable to detect results for the average waiting time for requirements in a queue in such QS. As is known from queuing theory, the average waiting time is the main characteristic for any QS. According to this characteristic, for example, are evaluated packet delays in packet-switched networks when simulating those using QS. The considered QS with ordinary and shifted input distributions are of type G/G/1.

In the queueing theory, the studies of G/G/1 systems are relevant because they are actively used in modern teletraffic theory; moreover, it is impossible to obtain solutions for such systems in the final form for the general case. The laws of the Weibull or Gamma distributions of the most general form, which provide a range of variation in the coefficients of variation from 0 to ∞, depending on the value of their parameters, do not allow them to be used in queuing theory. Therefore, it remains to use other particular laws of distributions.

In the study of G/G/1 systems, an important role is played by the method of spectral decomposition of the solution of the Lindley integral equation and most of the results in the theory of mass service are obtained using this method.

The object of study is the queueing systems type G/G/1.

The subject of study is the average waiting time in systems $E_2/HE_2/1$ and $E_2^- / HE_2^- /1$.

The purpose of the work is obtaining a solution for the average waiting time of requirements in the queue in closed form for the above-mentioned systems.

1 PROBLEM STATEMENT

The paper poses the problem of finding a solution for the waiting time of requirements in a queue in the $E_2/HE_2/1$ QS and $E_2^- / HE_2^- /1$. To solve the problem, it is necessary first to construct spectral decompositions for the indicated systems based on the theory of this method. Therefore, we give brief information about the method of spectral decomposition of the solution of LIE.

In presenting this method, we will adhere to the approach and symbolism of the author of the classical queuing theory [1]. We need to find the law of waiting time distribution in the system through the spectral decomposition of the form: $A^*(-s)\cdot B^*(s)-1=\psi_+(s)/\psi_-(s)$, where $\psi_+(s)$ and $\psi_-(s)$ are some rational functions of s that can be factorized. Functions $\psi_+(s)$ and $\psi_-(s)$ must satisfy the following conditions according to [1]:

1) for $\mathrm{Re}(s)>0$ function $\psi_+(s)$ is analytic without zeros in this half-plane;

2) for $\mathrm{Re}(s)<D$ function $\psi_-(s)$ is analytic without zeros in this half-plane, where D is some positive constant defined by the condition: $\lim\limits_{t\to\infty} a(t)/e^{-Dt}<\infty$. (1)

In addition, functions $\psi_+(s)$ and $\psi_-(s)$ must have the following properties:

$$\lim_{|s|\to\infty,\,\mathrm{Re}(s)>0}\frac{\psi_+(s)}{s}=1,\quad \lim_{|s|\to\infty,\,\mathrm{Re}(s)<D}\frac{\psi_-(s)}{s}=-1. \quad (2)$$

To solve the problem, it is necessary first to construct for these systems spectral decompositions of the form $A^*(-s)\cdot B^*(s)-1=\psi_+(s)/\psi_-(s)$, taking into account conditions (1), (2) in each case.

2 REVIEW OF THE LITERATURE

The method of spectral decomposition of the solution of the Lindley integral equation was first presented in detail in the classic queueing theory [1], and was subsequently used in many papers, including [7,8]. Another approach to solving the Lindley equation is used in Russian-language scientific literature [9]. In these papers, factorization was used instead of the term “spectral decomposition” and instead of functions $\psi_+(s)$ and $\psi_-(s)$ it used factorization components $\omega_+(z,t)$ and $\omega_-(z,t)$ of the function $1-z\cdot\chi(t)$, where $\chi(t)$ is the characteristic function of a random variable ξ with an arbitrary distribution function $C(t)$, and z is any number from the interval (−1, 1). This approach for obtaining results for systems under consideration is less convenient than the approach described and illustrated with numerous examples in [1].

In [2], for the first time, the results of an analysis of QS M/M/1 with shifted exponential distributions are presented. Article [3] presents the author’s results on systems with shifted hyperexponential and exponential distributions, in [4] – for the system with shifted hypererlangian distributions, in [5] – for the systems with shifted erlangian and exponential distributions, in [6] – for the systems with shifted hypererlangian and exponential distributions.

In [10] presents the results of the approach of queues to the Internet and mobile services as queues with a delay

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in time. It is shown that if information is delayed long enough, a Hopf bifurcation can occur, which can cause unwanted fluctuations in the queues.

At the same time, the scientific literature, including web-resources, the author was not able to detect results on the waiting time for the QS with Erlang and Hyper-Erlang input distributions of the second order of the general form. Approximate methods with respect to the laws of distributions are described in detail in [8, 13–16], and similar studies in queuing theory have recently been carried out in [17–24].

3 MATERIALS AND METHODS

For the $E_2/HE_2/1$ system, the distribution laws of the input flow intervals and the service time are given by the density functions of the form:

$$a(t)=4\lambda^2 te^{-2\lambda t}. \quad (3)$$

$$b(t)=4q\mu_1^2 te^{-2\mu_1 t}+4(1-q)\mu_2^2 te^{-2\mu_2 t}. \quad (4)$$

The solution for the average waiting time for the $E_2/HE_2/1$ system will be built based on the classical method of spectral decomposition of the solution of LIE, as shown in [2–7]. This approach allows us to determine not only the average waiting time, but also moments of higher orders of waiting time. Taking into account the definition of the variation of delay – jitter in telecommunications as the spread of waiting time from its average value [13], thereby we will be able to determine jitter through the dispersion of waiting time.

The Laplace transforms of the density functions (3) and (4) will be respectively:

$$A^*(s)=\left(\frac{2\lambda}{s+2\lambda}\right)^2;$$

$$B^*(s)=q\left(\frac{2\mu_1}{s+2\mu_1}\right)^2+(1-q)\left(\frac{2\mu_2}{s+2\mu_2}\right)^2.$$

Then the spectral decomposition $A^*(-s)\cdot B^*(s)-1=\psi_+(s)/\psi_-(s)$ of the solution of the IPL for the system $E_2/HE_2/1$ takes the form:

$$\frac{\psi_+(s)}{\psi_-(s)}=\left(\frac{2\lambda}{2\lambda-s}\right)^2\left[q\left(\frac{2\mu_1}{2\mu_1+s}\right)^2+(1-q)\left(\frac{2\mu_2}{2\mu_2+s}\right)^2\right]-1. \quad (5)$$

The expression standing in square brackets, we will present in the form:

$$\left[q\left(\frac{2\mu_1}{2\mu_1+s}\right)^2+(1-q)\left(\frac{2\mu_2}{2\mu_2+s}\right)^2\right]=$$

$$=\frac{q\left(16\mu_1^2\mu_2^2+16\mu_1^2\mu_2 s+4\mu_1^2 s^2\right)}{(2\mu_1+s)^2(2\mu_2+s)^2}+$$

$$+\frac{(1-q)\left(16\mu_1^2\mu_2^2+16\mu_1\mu_2^2 s+4\mu_2^2 s^2\right)}{(2\mu_1+s)^2(2\mu_2+s)^2}=$$

$$=\frac{b_0+b_1 s+b_2 s^2}{(2\mu_1+s)^2(2\mu_2+s)^2},$$

where are the intermediate parameters $b_0=16\mu_1^2\mu_2^2$, $b_1=16\mu_1\mu_2[q\mu_1+(1-q)\mu_2]$, $b_2=4[q\mu_1^2+(1-q)\mu_2^2]$. Continuing the decomposition, we get:

$$\frac{\psi_+(s)}{\psi_-(s)}=\frac{4\lambda^2(b_0+b_1 s+b_2 s^2)}{(2\lambda-s)^2(2\mu_1+s)^2(2\mu_2+s)^2}-$$

$$-\frac{(2\lambda-s)^2(2\mu_1+s)^2(2\mu_2+s)^2}{(2\lambda-s)^2(2\mu_1+s)^2(2\mu_2+s)^2}=$$

$$=\frac{-s(s^5-d_4 s^4-d_3 s^3-d_2 s^2-d_1 s-d_0)}{(2\lambda-s)^2(2\mu_1+s)^2(2\mu_2+s)^2}=$$

$$=\frac{-s(s+\sigma_1)(s+\sigma_2)(s+\sigma_3)(s+\sigma_4)(s-\sigma_5)}{(2\lambda-s)^2(2\mu_1+s)^2(2\mu_2+s)^2}.$$

Finally, the spectral decomposition of the LIE solution for the $E_2/HE_2/1$ system is:

$$\frac{\psi_+(s)}{\psi_-(s)}=\frac{-s(s+\sigma_1)(s+\sigma_2)(s+\sigma_3)(s+\sigma_4)(s-\sigma_5)}{(2\lambda-s)^2(2\mu_1+s)^2(2\mu_2+s)^2}. \quad (6)$$

The fifth degree polynomial in the numerator of the decomposition in the case of a stable system $\rho=\bar{\tau}_\mu/\bar{\tau}_\lambda<1$, provided

$$s^5-d_4 s^4-d_3 s^3-d_2 s^2-d_1 s-d_0 \quad (7)$$

with coefficients:

$$d_0=64\lambda\mu_1\mu_2[\mu_1\mu_2-\lambda(\mu_1+\mu_2)+4b_1\lambda^2],$$

$$d_1=16\{4\lambda\mu_1\mu_2(\mu_1+\mu_2)-\lambda^2[2\mu_1\mu_2+(\mu_1+\mu_2)^2]+4b_2\lambda^2\},$$

$$d_2=16\lambda[(\mu_1+\mu_2)^2+2\mu_1\mu_2]-16(\mu_1+\mu_2)(\lambda^2+\mu_1\mu_2),$$

$$d_3=-4[\lambda^2+\mu_1^2+\mu_2^2-4\lambda(\mu_1+\mu_2)+4\mu_1\mu_2],$$

$$d_4=4(\lambda-\mu_1-\mu_2)$$

has four real negative roots $-\sigma_1,-\sigma_2,-\sigma_3,-\sigma_4$ (or two real negative roots and two complexly conjugate with

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negative real parts) and one positive root σ_5. The study of the sign of the lower coefficient d_0 shows that it is always in the case of a stable system when $0<\rho<1$. Taking into account the minus sign in front d_0 of the polynomial (6), this also confirms the assumption about the presence of such roots of the polynomial. All the above results were obtained using the symbolic operations of Mathcad.

Further, taking into account conditions (1) and (2), we construct rational functions $\psi_+(s)$ and $\psi_-(s)$:

$$\psi_+(s)=\frac{s(s+\sigma_1)(s+\sigma_2)(s+\sigma_3)(s+\sigma_4)}{(2\mu_1+s)^2(2\mu_2+s)^2},$$

because the zeros of polynomial (6): $s=0$, $s=-\sigma_1$, $s=-\sigma_2$, $s=-\sigma_3$, $s=-\sigma_4$ and double poles $s=-2\mu_1$ and $s=-2\mu_2$ lie in the half-plane $\mathrm{Re}(s)\le 0$, $\psi_-(s)=-\frac{(2\lambda-s)^2}{(s-\sigma_5)}$, because its zeros and poles lie in the region $\mathrm{Re}(s)>D$ defined by condition (1). The fulfillment of conditions (1) and (2) for functions $\psi_+(s)$ and $\psi_-(s)$ is obvious, which is confirmed by Fig. 1.

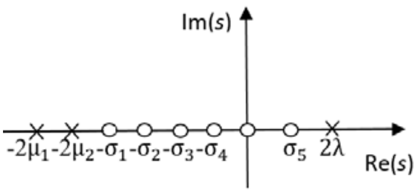

Fig.1. Zeros and poles of the function $\psi_+(s)/\psi_-(s)$ for the system $E_2/HE_2/1$

When constructing these functions, it is more convenient to mark the zeros and poles of the relation $\psi_+(s)/\psi_-(s)$ on the complex s – plane to eliminate errors in the construction of the functions $\psi_+(s)$ and $\psi_-(s)$. In Fig. 1, the poles are marked with crosses, and zeros are indicated by circles.

Further, according to the method of spectral decomposition, we find the constant K:

$$K=\lim_{s\to 0}\frac{\psi_+(s)}{s}=\lim_{s\to 0}\frac{(s+\sigma_1)(s+\sigma_2)(s+\sigma_3)(s+\sigma_4)}{(s+2\mu_1)^2(s+2\mu_2)^2}=$$
$$=\frac{\sigma_1\sigma_2\sigma_3\sigma_4}{16\mu_1^2\mu_2^2},$$

where $\sigma_1,\sigma_2,\sigma_3,\sigma_4$ – the absolute values of negative roots $-\sigma_1,-\sigma_2,-\sigma_3,-\sigma_4$. The constant K determines the probability that the demand entering the system finds it free.

Using the function and constant K, we define the Laplace transform of the PDF waiting time $W(y)$:

$$\Phi_+(s)=\frac{K}{\psi_+(s)}=\frac{\sigma_1\sigma_2\sigma_3\sigma_4}{16\mu_1^2\mu_2^2 s}\frac{(s+2\mu_1)^2(s+2\mu_2)^2}{(s+\sigma_1)(s+\sigma_2)(s+\sigma_3)(s+\sigma_4)}.$$

From here, the Laplace transform of the waiting time density function $W^*(s)=s\cdot\Phi_+(s)$ is

$$W^*(s)=\frac{\sigma_1\sigma_2\sigma_3\sigma_4(s+2\mu_1)^2(s+2\mu_2)^2}{16\mu_1^2\mu_2^2(s+\sigma_1)(s+\sigma_2)(s+\sigma_3)(s+\sigma_4)}. \quad (8)$$

To find the average waiting time, we find the derivative of the function $W^*(s)$ with a minus sign at the point s=0:

$$-\frac{dW^*(s)}{ds}\Big|_{s=0}=\frac{1}{\sigma_1}+\frac{1}{\sigma_2}+\frac{1}{\sigma_3}+\frac{1}{\sigma_4}-\frac{1}{\mu_1}-\frac{1}{\mu_2}.$$

Finally, the average wait time for the $E_2/HE_2/1$ system

$$\overline{W}=\frac{1}{\sigma_1}+\frac{1}{\sigma_2}+\frac{1}{\sigma_3}+\frac{1}{\sigma_4}-\frac{1}{\mu_1}-\frac{1}{\mu_2}. \quad (9)$$

From the expression (8), if necessary, you can also determine the moments of higher orders of the waiting time, for example, the second derivative of the transformation (8) at the point $s=0$ gives the second initial moment of the waiting time, which allows you to determine the dispersion of the waiting time, and hence jitter.

For the practical application of expression (9), it is necessary to determine the numerical characteristics of the distributions (3) E_2 and (4) HE_2.

Note that for the distribution of E_2: $\overline{\tau}_\lambda=\lambda^{-1}$, $c_\lambda=1/\sqrt{2}$. To find the numerical characteristics up to the second order for the distribution (4), we use the property of the Laplace transform $B^*(s)$ of the moment's reproduction and we write the initial moments:

$$\overline{\tau}_\mu=\frac{q}{\mu_1}+\frac{(1-q)}{\mu_2}, \quad (10)$$

$$\overline{\tau_\mu^2}=\frac{3}{2}\left[\frac{q}{\mu_1^2}+\frac{(1-q)}{\mu_2^2}\right]. \quad (11)$$

Considering equalities (10) and (11) as a record of the method of moments, we find the unknown distribution parameters μ_1,μ_2,q (3). The system of two equations (10), (11) is not predefined; therefore, we add to it an expression for the square of the coefficient of variation:

$$c_\mu^2=\frac{\overline{\tau_\mu^2}-(\overline{\tau}_\mu)^2}{(\overline{\tau}_\mu)^2}, \quad (12)$$

as a connecting condition between (10) and (11). In addition, the coefficient of variation will be used in the

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calculations as an input parameter of the system. Based on the form of equation (10), we set

$$\mu_1 = 2q / \bar{\tau}_\mu, \ \mu_2 = 2(1-q)/\bar{\tau}_\mu \qquad (13)$$

and demand the fulfillment of condition (12). Substituting the expressions (10), (11), (13) into (12), we obtain the fourth-degree equation for the parameter q: $q(1-q)[8(1+c_\mu^2)q^2 - 8(1+c_\mu^2)q + 3] = 0$. Rejecting the trivial solutions $q=0$ and $q=1$, we obtain a quadratic equation, having decided which, for uniqueness, we choose a larger root:

$$q = \frac{1}{2} + \sqrt{\frac{2(1+c_\mu^2)-3}{8(1+c_\mu^2)}}. \qquad (14)$$

From this it follows that the coefficient of variation $c_\mu > 1/\sqrt{2}$. Thus, a particular solution is obtained for an unspecified system of equations (10) and (11) by the selection method.

A similar approach to the approximation of the laws of distributions by the hyperexponential distribution was applied in the works of the author [3, 4]. Thus, the second-order hyperelang distribution law can be determined completely by the first two moments and cover the entire range of the coefficient of variation from $1/\sqrt{2}$ to ∞, which is wider than that the coefficient of variation for hyperexponential distribution $(1, \infty)$.

The quantities $\bar{\tau}_\lambda$, $\bar{\tau}_\mu$, c_λ, c_μ, defined above, will be considered as input parameters for calculating the average waiting time for the system $E_2/HE_2/1$. Then the calculation algorithm will be reduced to the sequential determination of the distribution parameters (4) from expressions (14), (13) and to finding the necessary roots of the polynomial (7), and then to using the formula (9).

Next, we consider a system that is fundamentally different from the QS studied. For the $E_2/HE_2/1$ system with shifted laws of distributions of input flow intervals and service time, these laws are defined by density functions of the form:

$$a(t) = \begin{cases} 4\lambda^2 (t-t_0) e^{-2\lambda(t-t_0)}, & t > t_0 \\ 0, & 0 \le t \le t_0 \end{cases}. \qquad (15)$$

$$b(t) = \begin{cases} 4q\mu_1^2 (t-t_0) e^{-2\mu_1(t-t_0)} + 4(1-q)\mu_2^2 (t-t_0) e^{-2\mu_2(t-t_0)}, & t > t_0 \\ 0, & 0 \le t \le t_0. \end{cases} \qquad (16)$$

Such a QS, unlike the conventional system, is denoted as $E_2^- / HE_2^- / 1$. In the work of the authors [2], a system with shifted exponential input distributions is designated as a system with a delay. The time shift of the exponential distribution transforms the classical M/M/1 system into a G/G/1 type system.

Statement. The spectral expansions $A^*(-s) \cdot B^*(s) - 1 = \psi_+(s)/\psi_-(s)$ of the LIE solution for systems $E_2^- / HE_2^- / 1$ and $E_2/HE_2/1$ completely coincide and have the form (5).

Proof. The Laplace transforms of functions (15) and (16) will be respectively:

$$A^*(s) = \left(\frac{2\lambda}{s+2\lambda}\right)^2 e^{-t_0 s},$$

$$B^*(s) = \left[q\left(\frac{2\mu_1}{s+2\mu_1}\right)^2 + (1-q)\left(\frac{2\mu_2}{s+2\mu_2}\right)^2 \right] \cdot e^{-t_0 s}.$$

The spectral decomposition $A^*(-s) \cdot B^*(s) - 1 = \frac{\psi_+(s)}{\psi_-(s)}$ of the LIE solution for the $E_2^- / HE_2^- / 1$ system will be:

$$\frac{\psi_+(s)}{\psi_-(s)} = \left(\frac{2\lambda}{2\lambda - s}\right)^2 e^{-t_0 s} \times$$
$$\times \left[q\left(\frac{2\mu_1}{2\mu_1 - s}\right)^2 + (1-q)\left(\frac{2\mu_2}{2\mu_2 - s}\right)^2 \right] e^{t_0 s} - 1 =$$
$$= \left(\frac{2\lambda}{2\lambda - s}\right)^2 \left[q\left(\frac{2\mu_1}{2\mu_1 - s}\right)^2 + (1-q)\left(\frac{2\mu_2}{2\mu_2 - s}\right)^2 \right] - 1.$$

Here, the exponential functions due to the opposite signs of the exponents are zeroed out and thus the shift operation is leveled. We thereby obtained the same expression (5). Further decomposition of the last expression will lead to the form (6), as was done for the system $E_2/HE_2/1$. Therefore, the spectral expansions for the systems $E_2^- / HE_2^- / 1$ and $E_2/HE_2/1$ completely coincide and have the form (5). The statement is proved.

Thus, considering the $E_2^- / HE_2^- / 1$ system, we can fully take advantage of the results obtained above for the $E_2/HE_2/1$ system, but with the changed numerical characteristics of the shifted distributions (15) and (16).

We define the numerical characteristics of the interval between the arrivals of requirements and service time for the new system $E_2^- / HE_2^- / 1$. To do this, we use the Laplace transforms of functions (15) and (16).

We define the numerical characteristics of the distribution (15). The average interval between the arrivals of requirements in the system is

$$\bar{\tau}_\lambda = \lambda^{-1} + t_0. \qquad (17)$$

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The second initial moment of this interval is $\left.\frac{d^2A^*(s)}{ds^2}\right|_{s=0}=\frac{3}{2\lambda^2}+2\frac{t_0}{\lambda}+t_0^2$, from where $\overline{\tau_\lambda^2}=\frac{3}{2\lambda^2}+2\frac{t_0}{\lambda}+t_0^2$. Define the square of the coefficient of variation $c_\lambda^2=\frac{\overline{\tau_\lambda^2}-(\overline{\tau_\lambda})^2}{(\overline{\tau_\lambda})^2}=\frac{1}{2(1+\lambda t_0)^2}$. Hence the coefficient of variation:

$$c_\lambda=[\sqrt{2}(1+\lambda t_0)]^{-1}. \qquad (18)$$

The value of the first derivative of the function $B^*(s)$ with a minus sign at the point s=0 is equal to

$$-\left.\frac{dB^*(s)}{ds}\right|_{s=0}=q\mu_1^{-1}+(1-q)\mu_2^{-1}+t_0 .$$

Hence, the average value of the intervals between adjacent requirements of the input flow will be equal to

$$\overline{\tau}_\mu=q\mu_1^{-1}+(1-q)\mu_2^{-1}+t_0 . \qquad (19)$$

The value of the second derivative of the function $B^*(s)$ at s=0 gives the second initial moment of service time

$$\overline{\tau_\mu^2}=t_0^2+2t_0[\frac{q}{\mu_1}+\frac{(1-q)}{\mu_2}]+\frac{3q}{2\mu_1^2}+\frac{3(1-q)}{2\mu_2^2}. \qquad (20)$$

From here, we define the square of the coefficient of variation of the arrival intervals:

$$c_\mu^2=\frac{\mu_1^2-2q\mu_2(\mu_1-\mu_2)+q(1-2q)(\mu_1-\mu_2)^2}{2[\mu_1-q(\mu_1-\mu_2)+t_0\mu_1\mu_2]^2}. \qquad (21)$$

Note that the coefficients of variation $0<c_\lambda<1/\sqrt{2}$ and $c_\mu>0$ for the shift parameter $t_0>0$. Thus, it is obvious that the $E_2^-/HE_2^-/1$ system is of type G/G/1.

Considering expressions (17) – (21) as a form of recording the method of moments, we find the unknown distribution parameters (15) and (16). We determine the distribution parameter (15) λ from (17) and get the value $\lambda=1/(\overline{\tau}_\lambda-t_0)$.

Finding distribution parameters (16) μ_1,μ_2,q will be similar to finding these parameters for distribution (4). Now, based on the form of equation (19), we set

$$\mu_1=2q/(\overline{\tau}_\mu-t_0)\,,\ \mu_2=2(1-q)/(\overline{\tau}_\mu-t_0) \qquad (22)$$

and demand the fulfillment of condition (21). Substituting solution (22) into (21), we solve the resulting equation of the fourth degree with respect to the parameter q, taking into account the condition $0<q<1$ and choose the desired solution

$$q=\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{3(\overline{\tau}_\mu-t_0)^2}{8[(\overline{\tau}_\mu-t_0)^2+c_\mu^2\overline{\tau}_\mu^2]}},$$

and then we determine the parameters μ_1 and μ_2 from (22).

By specifying the values $\overline{\tau}_\lambda$, $\overline{\tau}_\mu$, c_λ, c_μ, t_0 as the input parameters of the system, we thus determine all unknown parameters of the distributions (15) and (16) using the known method of moments.

Now consider the effect of the shift parameter on the coefficients of variation of the distributions. For the Erlang distribution of the intervals between arrivals, the coefficient of variation is $c_\lambda=1/\sqrt{2}$ and comparing with equality (18), we see that the shift parameter $t_0>0$ reduces this coefficient of variation $1+\lambda t_0$ times. For the conventional distribution of the HE_2 service time, as follows from expressions (10) – (12), we obtain

$$c_\mu^2=\frac{\mu_1^2-2q\mu_2(\mu_1-\mu_2)+q(1-2q)(\mu_1-\mu_2)^2}{2[\mu_1-q(\mu_1-\mu_2)]^2}.$$

Comparing the last expression with (21), we see that the time shift parameter $t_0>0$ reduces the coefficient of variation of the service time by $1+\frac{t_0\mu_1\mu_2}{[\mu_1(1-q)+\mu_2 q]}$ times. Taking into account the quadratic dependence of the average waiting time on the coefficients of variation of the arrival intervals and service time, we are convinced that the introduction of the shift parameter in the distribution laws reduces the average waiting time in the queue in the QS.

4 EXPERIMENTS

Below in the table 1 shows the calculation data for the $E_2/HE_2/1$ system and the $E_2/H_2/1$ system close to it for the cases of low, medium and high loads $\rho=0,1;0,5;0,9$. Note that the $E_2/HE_2/1$ system is applicable for $c_\lambda=1/\sqrt{2}$ and $c_\mu\ge 1/\sqrt{2}$, and the $E_2/H_2/1$ system for $c_\lambda=1/\sqrt{2}$ and $c_\mu\ge 1$. The load factor ρ in all tables is determined by the ratio of average intervals $\rho=\overline{\tau}_\mu/\overline{\tau}_\lambda$. The dashes in table 1 mean that for these input parameters the $E_2/H_2/1$ system is not applicable. The calculations given in all tables are carried out for the normalized service time $\overline{\tau}_\mu=1$.

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Table 1 – Results of experiments for QS $E_2/HE_2/1$and $E_2/H_2/1$

Input parameters		Average waiting time	
ρ	c_μ	For QS $E_2/HE_2/1$	For QS $E_2/H_2/1$
0.1	0.71	0.018	–
	2	0.160	0.160
	4	0.796	0.795
	8	3.452	3.448
0.5	0.71	0.395	–
	2	2.102	2.094
	4	8.092	8.082
	8	32.089	32.079
0.9	0.71	4.380	
	2	20.082	20.072
	4	74.075	74.065
	8	290.074	290.063

Tables 2 and 3 show the calculated values of the average waiting time for the $E_2^- / HE_2^- / 1$ system with delay also for cases of low, medium and high load $\rho = 0,1; 0,5; 0,9$ with the values of the shift parameter t_0 from 0.001 to 0.999 and the coefficients of service time variation $c_\mu = 0,71$ and $c_\mu = 2$, respectively, for the usual system $E_2/H_2/1$. The values of the parameters c_λ and c_μ in the $E_2^- / HE_2^- / 1$ system change according to expressions (18) and (21), respectively, as a result of the introduction of the shift parameter $t_0 > 0$, and decrease by $1+\lambda t_0$ and $1+\dfrac{t_0\mu_1\mu_2}{[\mu_1(1-q)+\mu_2 q]}$ times.

Table 2 – Results of experiments for QS $E_2^-/HE_2^-/1$ when for system $E_2/HE_2/1$ $c_\mu = 0,71$

Input parameters				Average waiting time	
ρ	c_λ	c_μ	t_0	For QS $E_2^- / HE_2^- / 1$	For QS $E_2/HE_2/1$
0.1	0.636	0.355	0.999	0.006	0.018
	0.672	0.473	0.5	0.006	
	0.700	0.645	0.1	0.013	
	0.706	0.703	0.01	0.017	
	0.707	0.709	0.001	**0.018**	
0.5	0.354	0.355	0.999	0.063	0.395
	0.530	0.473	0.5	0.124	
	0.672	0.645	0.1	0.321	
	0.704	0.703	0.01	0.386	
	0.707	0.709	0.001	**0.394**	
0.9	0.071	0.355	0.999	0.568	4.380
	0.389	0.473	0.5	1.511	
	0.643	0.645	0.1	3.578	
	0.701	0.703	0.01	4.292	
	0.706	0.709	0.001	**4.371**	

5 RESULTS

Comparison of the results for the two close systems in Table 1 confirms their very good agreement. As can be seen from tables 2 and 3, the average waiting time in the $E_2^-/HE_2^-/1$ system with increasing shift parameter decreases many times as compared with the conventional system $E_2/HE_2/1$.

Table 3 – Results of experiments for QS $E_2^-/HE_2^-/1$ when for system $E_2/HE_2/1$ $c_\mu = 2$

Input parameters				Average waiting time	
ρ	c_λ	c_μ	t_0	For QS $E_2^- / HE_2^- / 1$	For QS $E_2/HE_2/1$
0.1	0.636	1.001	0.999	0.064	0.160
	0.672	1.333	0.5	0.064	
	0.700	1.818	0.1	0.128	
	0.706	1.980	0.01	0.157	
	0.707	1.998	0.001	**0.160**	
0.5	0.357	1.001	0.999	0.500	2.102
	0.530	1.333	0.5	0.881	
	0.672	1.818	0.1	1.724	
	0.704	1.980	0.01	2.059	
	0.707	1.998	0.001	**2.098**	
0.9	0.071	1.001	0.999	4.504	20.082
	0.389	1.333	0.5	8.478	
	0.643	1.818	0.1	16.547	
	0.701	1.980	0.01	19.684	
	0.706	1.998	0.001	**20.041**	

Thus, Table 2 and 3 demonstrates the qualitative and quantitative influence of the shift parameter on the numerical characteristics of the distributions (15) and (16), and therefore, on the main characteristic of the system, the average waiting time. As one would expect, a decrease in the coefficients of variation c_λ and c_μ entails a decrease in the waiting time several times.

The data in the tables correlate well with the results of the method of two-moment approximation of the processes of receipts and services in the QS [11].

In the work, spectral expansions of the solution of the Lindley integral equation for two systems $E_2/HE_2/1$, $E_2^- / HE_2^- / 1$ are obtained, and it is proved that they completely coincide. Using the spectral decomposition, a formula is derived for the average waiting time in the queue for these systems in closed form.

The range of variation of the $E_2^- / HE_2^- / 1$ system parameters is much wider than that of the $E_2/HE_2/1$ system therefore these systems can be successfully applied in modern teletraffic theory.

6 DISCUSSION

As expected, the data table 2 and 3 fully confirm the above assumptions about the average waiting time in a system with a delay. In connection with the reduction of the coefficients of variation of the intervals of arrivals of requirements and the time of their maintenance due to the input of the shift parameter into the laws of distributions, the latency of requirements in the queue decreases in the system with delay. Moreover, this decrease is many times. In addition, with a decrease in the shift parameter t_0, the average waiting time in the system with delay tends to the value of this time in the conventional system, which further confirms the adequacy of the results obtained.

Thus, the constructed mathematical model of the $E_2^-/HE_2^-/1$ system with delay extends the range of

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applicability of the conventional $E_2/HE_2/1$ system by introducing the shift parameter $t_0 > 0$ into the distribution laws describing the functioning of the systems. As the results of computational experiments show, the constructed system with delay is qualitatively and quantitatively different from a conventional system. At the same time, the obtained calculation formulas for the average waiting time for these systems expand and supplement the incomplete formula of the queuing theory for G / G / 1 systems.

CONCLUSIONS

The article presents the solution to the problem of determining the average waiting time for two queuing systems $E_2/HE_2/1$ and $E_2^-/HE_2^-/1$ by the classical method of spectral decomposition.

The scientific novelty the obtained results consist in the fact that spectral expansions of the solution of the Lindley integral equation for the systems under consideration were obtained and with their help the formulas for the average waiting time in the queue for these systems in closed form were derived. These expressions extend and complement the well-known incomplete formula in queuing theory for the mean waiting time for systems of type G/G/1 with arbitrary laws of input flow distribution and service time.

The practical significance of the work lies in the fact that the obtained results can be successfully applied in the modern theory of teletraffic, where the delays of incoming traffic packets play a primary role. For this, it is necessary to know the numerical characteristics of the incoming traffic intervals and the service time at the level of the first two moments, which does not cause difficulties when using modern traffic analyzers.

Prospects for further research are seen in the continuation of the study of systems of type G/G/1 with other common input distributions and in expanding and supplementing the formulas for average waiting time.

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МОДЕЛЬ ТЕЛЕТРАФІКА НА ОСНОВІ СМО $E_2/HE_2/1$ ЗІ ЗВИЧАЙНИМИ ТА ЗСУНУТИМИ ВХІДНИМИ РОЗПОДІЛАМИ

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АНОТАЦІЯ

Актуальність. Актуальність дослідження систем G/G/1 пов'язана з їх затребуваністю в сучасній теорії телетрафіка і, отже, в теорії обчислювальних систем і мереж. У свою чергу, це випливає з того факту, що не можна отримати рішення для часу очікування для цих систем в кінцевому вигляді в загальному випадку при довільних законах розподілів вхідного потоку і часу обслуговування. Тому важливі дослідження таких систем для окремих випадків вхідних розподілів.

Мета роботи. Отримання рішення для основної характеристики системи – середнього часу очікування вимог в черзі для двох систем масового обслуговування типу G/G/1 зі звичайними і з зсунутими ерланговськими і гіперерлангівськими вхідними розподілами другого порядку.

Метод. Для вирішення поставленого завдання використаний класичний метод спектрального розкладання рішення інтегрального рівняння Линдли, який грає важливу роль в теорії систем G/G/1. Даний метод дозволяє отримати рішення для середнього часу очікування для розглянутих систем в замкнутій формі. Для практичного застосування отриманих результатів було використано відомий метод моментів теорії ймовірностей.

Результати. Вперше отримано спектральні розкладання рішення інтегрального рівняння Ліндлі для двох систем, за допомогою яких виведені розрахункові формули для середнього часу очікування в черзі в замкнутій формі. Система зі зсунутими ерланговськими і гіперерлангівськими вхідними розподілами забезпечує менший час очікування вимог в черзі в порівнянні зі звичайною системою за рахунок зменшення коефіцієнтів варіацій інтервалів між надходженнями вимог і часу обслуговування.

Висновки. Отримано спектральні розкладання рішення інтегрального рівняння Линдли для розглянутих систем і доведено їх повний збіг. Отже, збігаються і розрахункові вирази для середнього часу очікування в черзі для цих систем, але зі зміненими параметрами. Отримане розрахункове вираз розширює і доповнює відому незавершену формулу теорії масового обслуговування для середнього часу очікування для систем G/G/1. Такий підхід дозволяє розрахувати середній час очікування для зазначених систем в математичних пакетах для широкого діапазону зміни параметрів трафіку. Всі інші характеристики систем є похідними часу очікування.

Крім середнього часу очікування, такий підхід дає можливість також визначити моменти вищих порядків часу очікування. З огляду на той факт, що варіація затримки пакетів (джиттер) в телекомунікації визначається як дисперсія часу очікування від його середнього значення, то джиттер можна буде визначити через дисперсію часу очікування.

Отримані результати публікуються вперше.

КЛЮЧОВІ СЛОВА: ерлангівський та гіперерлангівський закони розподілу, інтегральне рівняння Ліндлі, метод спектрального розкладання, перетворення Лапласа.

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МОДЕЛЬ ТЕЛЕТРАФИКА НА ОСНОВЕ СМО $E_2/HE_2/1$ С ОБЫЧНЫМИ И СДВИНУТЫМИ ВХОДНЫМИ РАСПРЕДЕЛЕНИЯМИ

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АННОТАЦИЯ

Актуальность. Актуальность исследования систем G/G/1 связана с их востребованностью в современной теории телетрафика и, следовательно, в теории вычислительных систем и сетей. В свою очередь, это следует из того факта, что нельзя получить решения для времени ожидания для этих систем в конечном виде в общем случае при произвольных законах распределений входного потока и времени обслуживания. Поэтому важны исследования таких систем для частных случаев входных распределений.

Цель работы. Получение решения для основной характеристики системы – среднего времени ожидания требований в очереди для двух систем массового обслуживания типа G/G/1 с обычными и со сдвинутыми эрланговскими и гиперэрланговскими входными распределениями второго порядка.

Метод. Для решения поставленной задачи использован классический метод спектрального разложения решения интегрального уравнения Линдли, который играет важную роль в теории систем G/G/1. Данный метод позволяет получить решение для среднего времени ожидания для рассматриваемых систем в замкнутой форме. Для практического применения полученных результатов использован известный метод моментов теории вероятностей.

Результаты. Впервые получены спектральные разложения решения интегрального уравнения Линдли для двух систем, с помощью которых выведены расчетные выражения для среднего времени ожидания в очереди в замкнутой форме. Система со сдвинутыми эрланговскими и гиперэрланговскими входными распределениями обеспечивает меньшее время ожидания требований в очереди, по сравнению с обычной системой за счет уменьшения коэффициентов вариаций интервалов поступлений требований и времени обслуживания.

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Выводы. Получены спектральные разложения решения интегрального уравнения Линдли для рассматриваемых систем и доказано их полное совпадение. Следовательно, совпадают и расчетные выражения для среднего времени ожидания в очереди для этих систем, но с измененными параметрами. Полученное расчетное выражение расширяет и дополняет известную незавершенную формулу теории массового обслуживания для среднего времени ожидания для систем G/G/1. Такой подход позволяет рассчитать среднее время ожидания для указанных систем в математических пакетах для широкого диапазона изменения параметров трафика. Все остальные характеристики систем являются производными от времени ожидания. Кроме среднего времени ожидания, такой подход дает возможность определить и моменты высших порядков времени ожидания. Учитывая тот факт, что вариация задержки пакетов (джиттер) в телекоммуникациях определяется как разброс времени ожидания от его среднего значения, то джиттер можно будет определить через дисперсию времени ожидания.

Полученные результаты публикуется впервые.

КЛЮЧЕВЫЕ СЛОВА: эрланговский и гиперэрланговский законы распределения, интегральное уравнение Линдли, метод спектрального разложения, преобразование Лапласа.

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