UDC 621.391.1:621.395

RESEARCH OF TWO SYSTEMS $E_2/H_2/1$ WITH ORDINARY AND SHIFTED DISTRIBUTIONS BY THE SPECTRAL DECOMPOSITION METHOD

Tarasov V. N. – Dr. Sc., Professor, Head of Department of Software and Management in Technical Systems of Volga State University of Telecommunications and Informatics, Samara, Russian Federation.

ABSTRACT

Context. In the queueing theory, the studies of G/G/1 systems are relevant because it is impossible to obtain solutions for the waiting time in the final form in the general case with arbitrary laws of distributions of the input flow and of the service time. Therefore, the study of such systems for particular cases of input distributions is important. The problem of deriving a solution for the average waiting time in a queue in closed form for a pair of systems with ordinary and with shifted Erlang and hyperexponential input distributions is considered.

Objective. Obtaining a solution for the main system characteristic – the average waiting time in queue for two queuing systems of type G/G/1 with conventional and with shifted second-order Erlang and Hyperexponential input distributions.

Method. To solve this problem, we used the classical spectral decomposition method for solving the Lindley integral equation, which plays an important role in the theory of G/G/1 systems. This method allows obtaining a solution for the average waiting time for the considered systems in a closed form. For the practical application of the obtained results, the well-known probability theory moments method is used.

Results. For the first time, spectral expansions of the solution of the Lindley integral equation are obtained for two systems, with the help of which the formulas for the average waiting time in the queue are derived in closed form.

Conclusions. Spectral expansions of the solution of the Lindley integral equation for the systems under consideration are obtained and their complete coincidence is proved. Consequently, the formulas for the average waiting time in the queue for these systems are the same, but with modified parameters. It is shown that in the system with a delay in time, the average waiting time is less than in a conventional system. The resulting for waiting time formulas expand and supplement the known queuing theory incomplete formula for the average waiting time for G/G/1 systems with arbitrary laws distributions of input flow and service time. This approach allows us to calculate the average latency for these systems in mathematical packages for a wide range of traffic parameters. All other characteristics of the systems are derived from the waiting time. In addition to the average waiting time, such an approach makes it possible to determine also moments of higher orders of waiting time. Given the fact that the packet delay variation (jitter) in telecommunications is defined as the spread of the waiting time from its average value, the jitter can be determined through the variance of the waiting time. The results are published for the first time.

KEYWORDS: delayed system, $E_2/H_2/1$ system, the average waiting time, Laplace transform, the spectral decomposition method.

ABBREVIATIONS

LIE is a Lindley integral equation;
QS is a queuing system;
PDF is a probability distribution function.

NOMENCLATURE

$a(t)$ is a density function of the distribution of time between arrivals;

$A^*(s)$ is a Laplace transform of the function $a(t)$;

$b(t)$ is a density function of the distribution of service time;

$B^*(s)$ is a Laplace transform of the function $b(t)$;

c_λ is a coefficient of variation of time between arrivals;

c_μ is a coefficient of variation of service time;

E_2 is an erlangian distribution of the second order;

E_2^- is a shifted erlangian distribution of the second order;

H_2 is a hyperexponential distribution of the second order;

H_2^- is a shifted hyperexponential distribution of the second order;

G is an arbitrary distribution law;

M is an exponential distribution law;

$\bar{W}$ is an average waiting time in the queue;

$W^*(s)$ is a Laplace transform of waiting time density function;

λ is a parameters of the erlangian distribution law of the input flow;

μ is a parameters of the erlangian distribution law of service time;

μ_1, μ_2 is a parameters of the hyperexponential distribution law of service time;

ρ is a system load factor;

$\bar{\tau}_\lambda$ is an average time between arrivals;

$\bar{\tau}_\lambda^2$ is a second initial moment of time between arrivals;

$\bar{\tau}_\mu$ is an average service time;

$\overline{\tau_\mu^2}$ is a second initial moment of service time;

$\Phi_+(s)$ is a Laplace transform of the PDF of waiting time;

$\psi_+(s)$ is a first component of spectral decomposition;

DOI 10.15588/1607-3274-2020-3-6

$\psi_-(s)$ is a second component of spectral decomposition.

INTRODUCTION

This article is devoted to the analysis of $E_2/H_2/1$ QS with ordinary and with shifted erlangian; (E_2) and hyperexponential (H_2) input distributions. In [1], results are presented on the study of QS with time delay with shifted hyperexponential and exponential input distributions, obtained by the classical method of spectral expansion of the solution of the Lindley integral equation (LIE) [2–4]. In [1], it is shown that the average waiting time of a queue in the QS with a time lag is less than in the usual system with the same load factor due to the fact that the coefficients of variation of the arrivals c_λ and service times c_μ become less than one with the lag parameter $t_0 > 0$.

In this paper, based on the results of the above works, the method of spectral decomposition of the solution LIE is developed on the $E_2/H_2/1$ system. As a result, we have new QS with a delay, which is qualitatively different from the usual system. The considered QS with ordinary and shifted input distributions are of type G/G/1.

In the queueing theory, the studies of G/G/1 systems are relevant because they are actively used in modern teletraffic theory, moreover, one cannot obtain solutions for such systems in the final form for the general case. The laws of the Weibull or Gamma distributions of the most general form, which provide the range of variation of the coefficients of variation from 0 to ∞ depending on the value of their parameters, are not applicable in the spectral decomposition method. This is because the Laplace transform of the density function for these distributions cannot be expressed in elementary functions. Therefore, it is necessary to use other private laws of distributions.

In the study of G/G/1 systems, an important role is played by the method of spectral decomposition of the solution of the Lindley integral equation and most of the results in the theory of mass service are obtained using this method.

The object of study is the queueing systems type G/G/1.

The subject of study is the average waiting time in systems $E_2/H_2/1$ and $E_2^-/H_2^-/1$.

The purpose of the work is obtaining a solution for the average waiting time of requirements in the queue in closed form for the above-mentioned systems.

1 PROBLEM STATEMENT

The paper poses the problem of finding a solution for the waiting time of requirements in a queue in the $E_2/H_2/1$ and $E_2^-/H_2^-/1$ QS. To solve the problem, it is necessary first to construct spectral decompositions for the indicated systems based on the theory of this method. When using the method of spectral decomposition of a LIE solution, we will follow the approach and symbolism of the author of the classical queuing theory [2].

We need to find the law of waiting time distribution in the system through the spectral decomposition of the form $A^*(-s)\cdot B^*(s)-1=\psi_+(s)/\psi_-(s)$, where $\psi_+(s)$ and $\psi_-(s)$ are some rational functions of s that can be factorized. Functions $\psi_+(s)$ and $\psi_-(s)$ must satisfy specific conditions according to [2].

2 REVIEW OF THE LITERATURE

The method of spectral decomposition of the solution of the Lindley integral equation was first presented in detail in the classic queueing theory [2], and was subsequently used in many papers, including [3,4]. A different approach to solving Lindley's equation has been used in [10]. That work used factorization instead of the term "spectral decomposition" and instead of the functions $\psi_+(s)$ and $\psi_-(s)$ it used factorization components $\omega_+(z,t)$ and $\omega_-(z,t)$ of the function $1-z\cdot\chi(t)$, where $\chi(t)$ is the characteristic function of a random variable ξ with an arbitrary distribution function $C(t)$, and z is any number from the interval (−1, 1). This approach for obtaining results for systems under consideration is less convenient than the approach described and illustrated with numerous examples in [2].

In [1], the results on systems with delay $H_2/H_2/1$, $H_2/M/1$, $M/H_2/1$ are given, in [5] – on system with delay $HE_2/HE_2/1$, in [6] – on systems with a delay based on the QS $E_2/E_2/1$, $E_2/M/1$, $M/E_2/1$, and in [7] – on systems with a delay based on the QS $HE_2/M/1$. Article [9] presents the results for a system with a delay $M/HE_2/1$, and article [8] summarizes the results for eight systems with a delay in time.

In [11] presents the results of the approach of queues to the Internet and mobile services as queues with a delay in time. At the same time, the scientific literature, including web-resources, the author was not able to detect results on the waiting time for the QS with Erlang and Hyper exponential input distributions of the second order of the general form. Approximate methods with respect to the laws of distributions are described in detail in [4, 13–15], and similar studies in queuing theory have recently been carried out in [16–24].

3 MATERIALS AND METHODS

For the $E_2/H_2/1$ system, the distribution laws of the input flow intervals and the service time are given by the density functions of the form:

$$a(t)=4\lambda^2te^{-2\lambda t}. \quad (1)$$

$$b(t)=q\mu_1e^{-\mu_1t}+(1-q)\mu_2e^{-\mu_2t}. \quad (2)$$

We write the Laplace transform functions (1) and (2):

DOI 10.15588/1607-3274-2020-3-6

$$A^*(s)=\left(\frac{2\lambda}{2\lambda+s}\right)^2,\ B^*(s)=q\frac{\mu_1}{s+\mu_1}+(1-q)\frac{\mu_2}{s+\mu_2}.$$

The expression $A^*(-s)\cdot B^*(s)-1=\psi_+(s)/\psi_-(s)$ for the spectral decomposition of the solution of the LIE for the system $E_2/H_2/1$ takes the form:

$$\frac{\psi_+(s)}{\psi_-(s)}=\left(\frac{2\lambda}{2\lambda-s}\right)^2\times\left[q\frac{\mu_1}{\mu_1+s}+(1-q)\frac{\mu_2}{\mu_2+s}\right]-1= =\frac{-s(s+s_1)(s+s_2)(s-s_3)}{(2\lambda-s)^2(s+\mu_1)(s+\mu_2)}, \quad (3)$$

because the fourth-degree polynomial in the numerator of expression (3) can be represented as an expansion $-s(s^3-c_2s^2-c_1s-c_0)$ with coefficients

$$c_0=4\lambda^2 q(\mu_1-\mu_2)+4\lambda\mu_1(\mu_2-\lambda)],$$
$$c_1=4\lambda(\mu_1+\mu_2-\lambda)-\mu_1\mu_2,\ c_2=4\lambda-\mu_1-\mu_2.$$

In turn, the cubic polynomial

$$s^3-c_2s^2-c_1s-c_0 \quad (4)$$

with such coefficients it has two real negative roots $-s_1,-s_2$ and one positive root s_3 in the case of stationary mode, i.e. when $0<\rho=\overline{\tau}_\mu/\overline{\tau}_\lambda<1$. Based on the rules of the construction of functions $\psi_+(s)$ and $\psi_-(s)$, from the expression (3), we take the function $\psi_+(s)$

$$\psi_+(s)=\frac{s(s+s_1)(s+s_2)}{(s+\mu_1)(s+\mu_2)},$$

because the zeros $s=0$, $-s_1,-s_2$ of the polynomial (4), and the poles $s=-\mu_1$, $s=-\mu_2$ lie in the half-plane $\mathrm{Re}(s)\le 0$. For the function $\psi_-(s)$ from the expression (3) we take

$$\psi_-(s)=-\frac{(2\lambda-s)^2}{(s-s_3)},$$

because its zero $s=2\lambda$ and pole $s=s_3$ lie in the half-plane $\mathrm{Re}(s)\ge D$.

Now the fulfillment of conditions [2] for the constructed functions is obvious. This is confirmed by figure 1, where the zeros and the poles of the obtained decomposition (3) are shown on the complex s – plane. In Figure 1, the poles are marked with crosses, and zeros are indicated by circles.

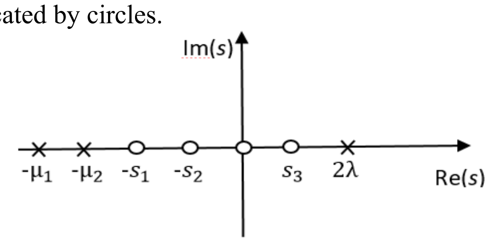

Figure 1 – Zeros and poles of the function $\psi_+(s)/\psi_-(s)$ for the system $E_2/H_2/1$

DOI 10.15588/1607-3274-2020-3-6

The constant K required to obtain the solution is $K=\lim\limits_{s\to 0}\frac{\psi_+(s)}{s}=\frac{s_1s_2}{\mu_1\mu_2}$. The constant K determines the probability that the demand entering the system finds it free.

Using the function and constant K, we define the Laplace transform of the PDF waiting time $W(y)$:

$$\Phi_+(s)=\frac{K}{\psi_+(s)}=\frac{s_1s_2(s+\mu_1)(s+\mu_2)}{s(s+s_1)(s+s_2)\mu_1\mu_2}.$$

From here, the Laplace transform of the waiting time density function $W^*(s)=s\cdot\Phi_+(s)$ is

$$W^*(s)=\frac{s_1s_2(s+\mu_1)(s+\mu_2)}{(s+s_1)(s+s_2)\mu_1\mu_2}. \quad (5)$$

To find the average waiting time, we find the derivative of the function $W^*(s)$ with a minus sign at the point s=0:

$$-\frac{dW^*(s)}{ds}|_{s=0}=-\frac{s_1s_2(s+\mu_1)(s+\mu_2)}{(s+s_1)(s+s_2)\mu_1\mu_2}|_{s=0}= =\frac{1}{s_1}+\frac{1}{s_2}-\frac{1}{\mu_1}-\frac{1}{\mu_2}.$$

Finally, the average wait time for the $E_2/H_2/1$ system

$$\overline{W}=\frac{1}{s_1}+\frac{1}{s_2}-\frac{1}{\mu_1}-\frac{1}{\mu_2}, \quad (6)$$

where s_1, s_2 the absolute values of the negative roots are $-s_1,-s_2$ of the cubic polynomial (4) with the coefficients given above, and μ_1, μ_2 – the distribution parameters (2). Thus, for the average waiting time in the QS $E_2/H_2/1$, the solution in closed form (6) is obtained.

From the expression (5), if necessary, you can also determine the moments of higher orders of the waiting time, for example, the second derivative of the transformation (5) at the point $s=0$ gives the second initial moment of the waiting time, which allows you to determine the dispersion of the waiting time, and hence jitter [12].

For the practical application of expression (6), it is necessary to determine the numerical characteristics of the distributions (1) E_2 and (2) H_2.

Note that for the distribution of E_2: $\overline{\tau}_\lambda=\lambda^{-1}$, $c_\lambda=1/\sqrt{2}$. This problem for the distribution law (2) using both the first two moments, and using the first three moments was considered in detail by the author in [4]. To do this, we write the expressions for the three initial moments of the distribution (2):

$$\overline{\tau}_\mu=\frac{q}{\mu_1}+\frac{(1-q)}{\mu_2},\ \overline{\tau_\mu^2}=\frac{2q}{\mu_1^2}+\frac{2(1-q)}{\mu_2^2},\ \overline{\tau_\mu^3}=\frac{6q}{\mu_1^3}+\frac{6(1-q)}{\mu_2^3}. \quad (7)$$

Then the square of the coefficient of variation of the service time will be equal to

$$c_\mu^2 = \frac{(1-q^2)\mu_1^2 - 2q(1-q)\mu_1\mu_2 + q(2-q)\mu_2^2}{[(1-q)\mu_1 + q\mu_2]^2}. \quad (8)$$

In this case, to determine the unknown parameters using the first two moments in [4], the following expressions were obtained

$$\mu_1 = 2q/\overline{\tau}_\mu, \; \mu_2 = 2(1-q)/\overline{\tau}_\mu,$$

$$q = \frac{1}{2}[1 \pm \sqrt{(c_\mu^2 - 1)/(c_\mu^2 + 1)}].$$

In this case, for the probability q you can take any of these values. It follows that the coefficient of variation $c_\mu \geq 1$. It now remains to determine the values of the desired roots $-s_1$, $-s_2$ polynomial (4) to use formula (6) for the given input parameters.

When approximating using the first three moments, in order to find the distribution parameters (2), it is necessary in the Mathcad package to solve the system of three equations (7) obtained by the method of moments. In this case, a necessary and sufficient condition for the existence of a solution is the fulfillment of the condition: $\overline{\tau_\lambda^3} \cdot \overline{\tau_\lambda} \geq 1,5\overline{\tau_\lambda^2}$ [13].

Next, we consider a system that is fundamentally different from the QS studied. For the $E_2/H_2/1$ system with shifted laws of distributions of input flow intervals and service time, these laws are defined by density functions of the form:

$$a(t) = \begin{cases} 4\lambda^2 (t - t_0) e^{-2\lambda(t-t_0)}, \; t > t_0, \\ 0, \; 0 \leq t \leq t_0, \end{cases}. \quad (9)$$

$$b(t) = \begin{cases} q\mu_1 e^{-\mu_1(t-t_0)} + (1-q)\mu_2 e^{-\mu_2(t-t_0)}, \; t > t_0, \\ 0, \; 0 \leq t \leq t_0. \end{cases} \quad (10)$$

Such a QS, unlike the conventional system, is denoted as $E_2^- / H_2^- / 1$.

Statement. The spectral expansions $A^*(-s) \cdot B^*(s) - 1 = \psi_+(s)/\psi_-(s)$ of the LIE solution for systems $E_2^- / H_2^- / 1$ and $E_2/H_2/1$ completely coincide and have the form (3). Consequently, the Laplace transforms of the waiting time density function for them also coincide.

Proof. The Laplace transforms of functions (9) and (10) will be respectively:

$$A^*(s) = \left(\frac{2\lambda}{s + 2\lambda}\right)^2 e^{-t_0 s},$$

$$B^*(s) = [q\frac{\mu_1}{s+\mu_1} + (1-q)\frac{\mu_2}{s+\mu_2}]e^{-t_0 s}.$$

The spectral decomposition $A^*(-s) \cdot B^*(s) - 1 = \psi_+(s)/\psi_-(s)$ of the LIE solution for the $E_2^- / H_2^- / 1$ system will be:

$$\frac{\psi_+(s)}{\psi_-(s)} = \left(\frac{2\lambda}{2\lambda - s}\right)^2 e^{t_0 s} \times [q\frac{\mu_1}{s+\mu_1} + (1-q)\frac{\mu_2}{s+\mu_2}]e^{-t_0 s} - 1 =$$
$$= \frac{-s(s+s_1)(s+s_2)(s-s_3)}{(2\lambda - s)^2 (s+\mu_1)(s+\mu_2)}.$$

Here, the exponential functions due to the opposite signs of the exponents are zeroed out and thus the shift operation is leveled. We thereby obtained the same expression (3). Therefore, the spectral expansions for the $E_2^- / H_2^- / 1$ and $E_2/H_2/1$ system completely coincide and have the form (3). Thus, all the above considerations for the $E_2/H_2/1$ system are also valid for the system, but already with the changed numerical characteristics of the shifted distributions (7) and (8). The statement is proved.

Thus, considering the $E_2^- / H_2^- / 1$ system, we can fully take advantage of the results obtained above for the $E_2/H_2/1$ system, but with the changed numerical characteristics of the shifted distributions (9) and (10).

We define the numerical characteristics of the interval between the arrivals of requirements and service time for the new $E_2^- / H_2^- / 1$ system. To do this, we use the Laplace transforms of functions (9) and (10).

Now we write the equations for the first two initial moments for determining the unknown distribution parameters (9):

$$\overline{\tau}_\lambda = \lambda^{-1} + t_0, \quad (11)$$

$$\overline{\tau_\lambda^2} = t_0^2 + \frac{2t_0}{\lambda} + \frac{3}{2\lambda^2}. \quad (12)$$

Define the square of the coefficient of variation of the interval between the arrivals of requirements $c_\lambda^2 = \overline{\tau_\lambda^2}/(\overline{\tau_\lambda})^2 - 1 = 1/2(1+\lambda t_0)^2$.

Hence the coefficient of variation:

$$c_\lambda = [\sqrt{2}(1 + \lambda t_0)]^{-1}. \quad (13)$$

The value of the first derivative of the function $B^*(s)$ with a minus sign at the point s=0 is equal to

$$-\left.\frac{dB^*(s)}{ds}\right|_{s=0} = q\mu_1^{-1} + (1-q)\mu_2^{-1} + t_0.$$

Hence, the average service time will be equal to

$$\overline{\tau}_\mu = q\mu_1^{-1} + (1-q)\mu_2^{-1} + t_0. \quad (14)$$

The value of the second derivative of the function $B^*(s)$ at s=0 gives the second initial moment of service time $\overline{\tau_\mu^2} = 2[q\mu_1^{-2} + (1-q)\mu_2^{-2}] + t_0^2 + 2t_0[q\mu_1^{-1} + (1-q)\mu_2^{-1}]$.

From here we define the square of the coefficient of variation of the service time:

DOI 10.15588/1607-3274-2020-3-6

$$c_\mu^2 = \frac{[(1-q^2)\mu_1^2 - 2\mu_1\mu_2 q(1-q) + q(2-q)\mu_2^2]}{[t_0\mu_1\mu_2 + (1-q)\mu_1 + q\mu_2]^2}. \quad (15)$$

Note that the coefficients of variation $0 < c_\lambda < 1/\sqrt{2}$ and $c_\mu > 0$ for the shift parameter $t_0 > 0$.

Considering expressions (11), (13) and (14) (15) as a form of recording the method of moments, we find the unknown distribution parameters (9) and (10). We determine the distribution parameter (9) λ from (11) and get the value $\lambda = 1/(\bar{\tau}_\lambda - t_0)$.

Finding distribution parameters (10) μ_1, μ_2, q will be similar to finding these parameters for distribution (2). Now, based on the form of equation (14), we set

$$\mu_1 = 2q/(\bar{\tau}_\mu - t_0), \; \mu_2 = 2(1-q)/(\bar{\tau}_\mu - t_0) \quad (16)$$

and demand the fulfillment of condition (15). Substituting the particular solution (16) into equality (15) and eliminating the trivial solutions q=0, q=1 from the 4th degree equation $q^2 - q + \frac{(\bar{\tau}_\mu - t_0)^2}{2[(\bar{\tau}_\mu - t_0)^2 + c_\mu^2 \bar{\tau}_\mu^2]} = 0$ with respect to q, we obtain the solution for probability q as the roots of the quadratic equation:

$$q = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{(\bar{\tau}_\mu - t_0)^2}{2[(\bar{\tau}_\mu - t_0)^2 + c_\mu^2 \bar{\tau}_\mu^2]}}, \quad (17)$$

and then we determine the parameters μ_1 and μ_2 from (16). In this case, as the parameter q, you can choose any of the two values. Consequently, the range of applicability of the system $E_2^- / H_2^- / 1$ will be determined by the nonnegativity of the expression under the square root.

From expression (17) it follows that the input parameters μ, c_μ, t_0 are constrained $c_\mu \geq 1 - t_0/\bar{\tau}_\mu$, and in turn, from (17) it follows that $0 < t_0 < \bar{\tau}_\mu$. Thus, the $E_2^- / H_2^- / 1$ system is applicable when performing constraints

$$c_\mu \geq 1 - t_0/\bar{\tau}_\mu, \; 0 < t_0 < \bar{\tau}_\mu. \quad (18)$$

Let us now estimate the effect of the shift parameter t_0 on the coefficient of variation of the service time c_μ.

Comparison of expressions (8) and (15) shows that c_μ for distribution (10) decreases by $1 + t_0\mu_1\mu_2/[\mu_1(1-q) + \mu_2 q]$ times.

By specifying the values $\bar{\tau}_\lambda$, $\bar{\tau}_\mu$, c_λ, c_μ, t_0 as the input parameters of the system, we thus determine all unknown parameters of the distributions (9) and (10) using the known method of moments. Further, having determined the absolute values of the negative roots s_1, s_2 of the cubic polynomial (4), we can calculate the average waiting time by expression (6) for the ranges of variation of the coefficients of variation $c_\lambda \in (0, 1/\sqrt{2})$ and $c_\mu \in (0, \infty)$ depending on the value of the shift parameter $t_0 > 0$.

DOI 10.15588/1607-3274-2020-3-6

4 EXPERIMENTS

Below in the table. 1 shows the calculation data for the $E_2/H_2/1$ system for the cases of low, medium and high loads $\rho = 0,1; 0,5; 0,9$. Note that the $E_2/H_2/1$ system is applicable for $c_\lambda = 1/\sqrt{2}$, $c_\mu \geq 1$. The load factor ρ in all tables is determined by the ratio of average intervals $\rho = \bar{\tau}_\mu / \bar{\tau}_\lambda$. The calculations given in all tables are carried out for the normalized service time $\bar{\tau}_\mu = 1$.

Table 1 – Results of experiments for QS $E_2/H_2/1$

Input parameters ρ, c_μ	Average waiting time for QS $E_2/H_2/1$			
ρ	c_μ =1	c_μ =2	c_μ =4	c_μ =8
0.1	0.030	0.160	0.795	3.448
0.5	0.618	2.094	8.082	32.079
0.9	6.588	20.072	74.065	290.063

Tables 2 and 3 show the calculation data for the system with delay also for the cases of low, medium and high load $\rho = 0,1; 0,5; 0,9$ for the values $c_\mu = 1$ and $c_\mu = 2$, accordingly, for the conventional system $E_2/H_2/1$ with the values of the shift parameter t_0 from 0,001 to 0,99 for the system with a delay.

Table 2 – Results of experiments for QS $E_2^-/H_2^-/1$ when for system $E_2/H_2/1$ $c_\mu = 1$

Input parameters				Average waiting time	
ρ	c_λ	c_μ	t_0	For QS $E_2^- / H_2^- / 1$	For QS $E_2/H_2/1$
0.1	0.637	0.503	0.99	0.011	0.030
	0.672	0.667	0.5	0.011	
	0.700	0.909	0.1	0.023	
	0.706	0.990	0.01	0.029	
	0.707	0.999	0.001	**0.030**	
0.5	0.357	0.503	0.99	0.125	0.618
	0.530	0.667	0.5	0.223	
	0.672	0.909	0.1	0.499	
	0.704	0.990	0.01	0.605	
	0.707	0.999	0.001	**0.617**	
0.9	0.077	0.503	0.99	1.135	6.588
	0.389	0.667	0.5	2.480	
	0.643	0.909	0.1	5.396	
	0.701	0.990	0.01	6.457	
	0.707	0.999	0.001	**6.575**	

Table 3 – Results of experiments for QS $E_2^-/H_2^-/1$ when for system $E_2/HE_2/1$ $c_\mu = 2$

Input parameters				Average waiting time	
ρ	c_λ	c_μ	t_0	For QS $E_2^-/H_2^-/1$	For QS $E_2/H_2/1$
0.1	0.637	1.005	0.99	0.055	0.160
	0.672	1.333	0.5	0.065	
	0.700	1.818	0.1	0.128	
	0.706	1.980	0.01	0.156	
	0.707	1.998	0.001	**0.159**	
0.5	0.357	1.005	0.99	0.504	2.094
	0.530	1.333	0.5	0.877	
	0.672	1.818	0.1	1.716	
	0.704	1.980	0.01	2.051	
	0.707	1.998	0.001	**2.089**	
0.9	0.077	1.005	0.99	4.544	20.072
	0.389	1.333	0.5	8.473	
	0.643	1.818	0.1	16.538	
	0.701	1.980	0.01	19.674	
	0.706	1.998	0.001	**20.031**	

5 RESULTS

In the work, spectral expansions of the solution of the Lindley integral equation for two systems $E_2/H_2/1$, $E_2^-/H_2^-/1$ are obtained, and it is proved that they completely coincide. Using the spectral decomposition, a formula is derived for the average waiting time in the queue for these systems in closed form. These formulas complement and extend the well-known incomplete formula for the average waiting time for G/G/1 systems.

The operation of the shift in time on the one hand, leads to an increase in system load with a delay. For example, for a $E_2^-/H_2^-/1$ system with a delay, the load is increased by $\frac{1+\mu_1\mu_2 t_0/[q\mu_2+(1-q)\mu_1]}{(1+\lambda t_0)}$ compared to the usual system $E_2/H_2/1$. The time shift operation, on the other hand, reduces the variation coefficients of the interval between arrivals and the service time of requirements. Because the average waiting time in the G/G/1 system is related to the coefficients of variation of the arrival intervals and service time by the quadratic dependence, the average waiting time in the delayed system will be less than in a conventional system with the same load factor.

For example, for a $E_2^-/H_2^-/1$ system with a load ρ =0.9 and a shift parameter t_0=0,99, the coefficient of variation of the arrival intervals c_λ decreases with for the usual system $E_2/H_2/1$ to 0.077 for a QS $E_2^-/H_2^-/1$. The service time variation coefficient decreases from 2 to 1,005, and the waiting time decreases from 6.59-time units for a conventional system to almost 1.14-time units for a latency system, i.e. almost 6 times (Table 2). The situation is similar with the results of Table 3.

The range of variation of the $E_2^-/H_2^-/1$ system parameters is much wider than that of the $E_2/H_2/1$ system therefore; these systems can be successfully applied in modern teletraffic theory.

6 DISCUSSION

As can be seen from tables 2 and 3, the average waiting time in the $E_2^-/H_2^-/1$ system with increasing shift parameter decreases many times as compared with the conventional system $E_2/H_2/1$.

As expected, the data table 2 and 3 fully confirm the above assumptions about the average waiting time in a system with a delay. In connection with the reduction of the coefficients of variation of the intervals of arrivals of requirements and the service time due to the input of the shift parameter into the laws of distributions, the latency of requirements in the queue decreases in the system with a delay. Moreover, this decrease is many times. In addition, with a decrease in the shift parameter t_0, the average waiting time in the system with delay tends to the value of this time in the conventional system, which further confirms the adequacy of the results obtained.

Thus, table 2 and 3 demonstrates the qualitative and quantitative influence of the shift parameter on the numerical characteristics of distributions (11) and (12), as well as on the main characteristic of the system – the average waiting time.

CONCLUSIONS

The article presents the solution to the problem of determining the average waiting time for two queuing systems $E_2/H_2/1$ and $E_2^-/H_2^-/1$ by the classical method of spectral decomposition.

The scientific novelty the obtained results consist in the fact that spectral expansions of the solution of the Lindley integral equation for the systems under consideration were obtained and with their help the formulas for the average waiting time in the queue for these systems in closed form were derived. These expressions extend and complement the well-known incomplete formula in queuing theory for the mean waiting time for systems of type G/G/1 with arbitrary laws of input flow distribution and service time.

The practical significance of the work lies in the fact that the obtained results can be successfully applied in the modern theory of teletraffic, where the delays of incoming traffic packets play a primary role. For this, it is necessary to know the numerical characteristics of the incoming traffic intervals and the service time at the level of the first two moments, which does not cause difficulties when using modern traffic analyzers.

Prospects for further research are seen in the continuation of the study of systems of type G/G/1 with other common input distributions and in expanding and supplementing the formulas for average waiting time.

ACKNOWLEDGEMENTS

This work was carried out as part of the author's scientific school “Methods and Models for the Research of Computing Systems and Networks”, registered at the

DOI 10.15588/1607-3274-2020-3-6

Russian Academy of Natural Sciences on 31.03.2015 and was supported by the University of PSUTI.

REFERENCES

1. Kleinrock L. Queueing Systems, Vol. I: Theory. New York, Wiley, 1975, 417 p.
2. Brannstrom N. A. Queueing Theory analysis of wireless radio systems. Appllied to HS-DSCH. Lulea university of technology, 2004, 79 p.
3. Whitt W. Approximating a point process by a renewal process: two basic methods, *Operation Research*, 1982, Vol. 30, No. 1, pp. 125–147.
4. Tarasov V. N. Extension of the Class of Queueing Systems with Delay, *Automation and Remote Control*, 2018, Vol. 79, No. 12, pp. 2147–2157. DOI: 10.1134/S0005117918120056.
5. Tarasov V.N. Analysis and comparison of two queueing systems with hypererlangian input distributions, *Radio Electronics, Computer Science, Control*, 2018, Vol. 47, No. 4, pp. 61–70. DOI 10.15588/1607-3274-2018-4-6.
6. Tarasov V.N., Bakhareva N.F. Research of queueing systems with shifted erlangian and exponential input distributions, *Radio Electronics, Computer Science, Control*, 2019, Vol. 48, No. 1, pp. 67–76. DOI 10.15588/1607-3274-2019-1-7.
7. Tarasov V.N. The analysis of two queuing systems $HE_2/M/1$ with ordinary and shifted input distributions, *Radio Electronics, Computer Science, Control*, 2019, Vol. 49, No. 2, pp. 71–79. DOI: 10.15588/1607-3274-2019-2-8.
8. Tarasov V.N. Queueing systems with delay. *Radio Electronics, Computer Science, Control*, 2019, Vol. 50, No. 3, pp. 71–79. DOI: 10.15588/1607-3274-2019-4-5.
9. Tarasov V.N., Bakhareva N.F. Comparative analysis of two Queuing Systems $M/HE_2/1$ with ordinary and with the shifted input Distributions, *Radio Electronics, Computer Science, Control*, 2019, Vol. 51, No. 4, pp. 50–58. DOI 10.15588/1607-3274-2019-4-5.
10. Bocharov P. P., Pechinkin A. V. Teoriya massovogo obsluzhivaniya. Moscow, Publishing House of Peoples' Friendship University, 1995, 529 p.
11. Novitzky S., Pender J., Rand R. H., Wesson E. Nonlinear Dynamics in Queueing Theory: Determining the Size of Oscillations in Queues with Delay, *SIAM J. Appl. Dyn. Syst.*, 18–1 2019, Vol. 18, No. 1, pp. 279–311. DOI: https://doi.org/10.1137/18M1170637.
12. RFC 3393 IP Packet Delay Variation Metric for IP Performance Metrics (IPPM) [Electronic resource]. Available at: https://tools.ietf.org/html/rfc3393.
13. Myskja A. An improved heuristic approximation for the GI/GI/1 queue with bursty arrivals. Teletraffic and datatraffic in a Period of Change. ITC-13. Elsevier Science Publishers, 1991, pp. 683–688.
14. Liu X. Diffusion approximations for double-ended queues with reneging in heavy traffic, *Queueing Systems*: *Theory and Applications*, Springer, 2019, Vol. 91, No. 1, pp. 49–87. DOI: 10.1007/s11134-018-9589-7.
15. Poojary S., Sharma V. An asymptotic approximation for TCP CUBIC, *Queueing Systems*: *Theory and Applications*, 2019, Vol. 91, No. 1, pp. 171–203. DOI: 10.1007/s11134-018-9594-x.
16. Aras A.K., Chen X. & Liu Y. Many-server Gaussian limits for overloaded non-Markovian queues with customer abandonment, *Queueing Systems*, 2018, Vol. 89, No. 1, pp. 81–125. DOI: https://doi.org/10.1007/s11134-018-9575-0.
17. Jennings O.B., Pender J. Comparisons of ticket and standard queues, *Queueing Systems*, 2016, Vol. 84, No. 1, pp. 145–202. DOI: https://doi.org/10.1007/s11134-016-9493-y.
18. Gromoll H. C., Terwilliger B., Zwart B. Heavy traffic limit for a tandem queue with identical service times, *Queueing Systems*, 2018, Vol. 89, No. 3, pp. 213–241. DOI: https://doi.org/10.1007/s11134-017-9560-z.
19. Legros B. M/G/1 queue with event-dependent arrival rates. *Queueing Systems*, 2018, Vol. 89, No. 3, pp. 269–301. DOI: https://doi.org/10.1007 /s11134-017-9557-7/.
20. Bazhba M., Blanchet J., Rhee CH., et al. Queue with heavy-tailed Weibull service times, *Queueing Systems*, 2019, Vol. 93, No. 11, pp. 1–32. https://doi.org/10.1007/s11134-019-09640-z/
21. Adan I., D'Auria B., Kella O. Special volume on 'Recent Developments in Queueing Theory' of the third ECQT conference. *Queueing Systems*, 2019, Vol. 93, No. 1, pp. 1–190. DOI: https://doi.org/10.1007/s11134-019-09630-1.
22. Adan I., D'Auria B., Kella O. Special volume on 'Recent Developments in Queueing Theory' of the third ECQT conference: part 2, *Queueing Systems*, 2019, pp. 1–2. DOI: https://doi.org/10.1007/s11134-019-09637-8.
23. Tibi D. Martingales and buffer overflow for the symmetric shortest queue model. *Queueing Systems,* Vol. 93, 2019, pp. 153–190. DOI:10.1007/s11134-019-09628-9.
24. Jacobovic R., Kella O. Asymptotic independence of regenerative processes with a special dependence structure. *Queueing Systems,* 2019, Vol. 93, pp. 139–152. DOI:10.1007/s11134-019-09606-1.

Received 18.03.2020.
Accepted 25.06.2020.

УДК 621.391.1: 621.395

ДОСЛІДЖЕННЯ ДВОХ СИСТЕМ $E_2/H_2/1$ ЗІ ЗВИЧАЙНИМИ ТА ЗСУНУТИМИ РОЗПОДІЛАМИ МЕТОДОМ СПЕКТРАЛЬНОГО РОЗКЛАДАННЯ

Тарасов В. Н. – д-р техн. наук, професор, завідувач кафедри програмного забезпечення та управління в технічних системах Поволзького державного університету телекомунікацій та інформатики, Россия.

АНОТАЦІЯ

Актуальність. В теорії масового обслуговування дослідження систем G/G/1 актуальні в зв'язку з тим, що не можна отримати рішення для часу очікування в кінцевому вигляді в загальному випадку при довільних законах розподілів вхідного потоку і часу обслуговування. Тому важливі дослідження таких систем для окремих випадків вхідних розподілів. Була розглянута задача виведення рішення для середнього часу очікування в черзі в замкнутій формі для двох систем зі звичайними і зі зсунутими ерлангівськими та гіперекспонентними вхідними розподілами.

DOI 10.15588/1607-3274-2020-3-6

Мета роботи. Отримання рішення для основної характеристики системи – середнього часу очікування вимог в черзі для двох систем масового обслуговування типу G/G/1 зі звичайними та зі зсунутими ерлангівськими та гіперекспонентними вхідними розподілами.

Метод. Для вирішення поставленого завдання був використаний класичний метод спектрального розкладання рішення інтегрального рівняння Ліндлі. Цей метод дозволяє отримати рішення для середнього часу очікування для розглянутих систем в замкнутій формі. Метод спектрального розкладання рішення інтегрального рівняння Ліндлі грає важливу роль в теорії систем G/G/1. Для практичного застосування отриманих результатів було використано відомий метод моментів теорії ймовірностей.

Результати. Вперше отримано спектральне розкладання рішення інтегрального рівняння Линдли для двох систем, за допомогою якого виведено розрахункове вираз для середнього часу очікування в черзі в замкнутій формі.

Висновки. Отримано спектральне розкладання рішення інтегрального рівняння Ліндлі для розглянутих систем, та з їх допомогою виведено розрахункове вираз для середнього часу очікування в черзі для цих систем в замкнутій формі. Показано, що в системі з запізненням у часі середній час очікування менше, ніж у звичайній системі. Отримане розрахункове вираз для часу очікування розширює і доповнює відому незавершену формулу теорії масового обслуговування для середнього часу очікування для систем G/G/1 з довільними законами розподілів вхідного потоку і часу обслуговування. Такий підхід дозволяє розрахувати середній час очікування для зазначених систем в математичних пакетах для широкого діапазону зміни параметрів трафіку. Всі інші характеристики систем є похідними часу очікування.

Крім середнього часу очікування, такий підхід дає можливість також визначити моменти вищих порядків часу очікування. З огляду на той факт, що варіація затримки пакетів (джиттер) в телекомунікації визначається як дисперсія часу очікування від його середнього значення, то джиттер можна буде визначити через дисперсію часу очікування.

Отримані результати публікуються вперше.

КЛЮЧОВІ СЛОВА: система з запізненням, система $E_2/H_2/1$, перетворення Лапласа, середній час очікування в черзі, метод спектрального розкладання.

УДК 621.391.1: 621.395

ИССЛЕДОВАНИЕ ДВУХ СИСТЕМ $E_2/H_2/1$ С ОБЫЧНЫМИ И СО СДВИНУТЫМИ РАСПРЕДЕЛЕНИЯМИ МЕТОДОМ СПЕКТРАЛЬНОГО РАЗЛОЖЕНИЯ

Тарасов В. Н. – д-р техн. наук, профессор, заведующий кафедрой программного обеспечения и управления в технических системах Поволжского государственного университета телекоммуникаций и информатики, Росія.

АННОТАЦИЯ

Актуальность. В теории массового обслуживания исследования систем G/G/1 актуальны в связи с тем, что нельзя получить решения для времени ожидания в конечном виде в общем случае при произвольных законах распределений входного потока и времени обслуживания. Поэтому важны исследования таких систем для частных случаев входных распределений. Рассмотрена задача вывода решения для среднего времени ожидания в очереди в замкнутой форме для пары систем с обычными и со сдвинутыми эрланговскими и гиперэкспоненциальными входными распределениями.

Цель работы. Получение решения для основной характеристики систем – среднего времени ожидания требований в очереди для двух систем массового обслуживания типа G/G/1 с обычными и со сдвинутыми гиперэкспоненциальными и эрланговскими входными распределениями.

Метод. Для решения поставленной задачи использован классический метод спектрального разложения решения интегрального уравнения Линдли. Данный метод позволяет получить решение для среднего времени ожидания для рассматриваемых систем в замкнутой форме. Метод спектрального разложения решения интегрального уравнения Линдли играет важную роль в теории систем G/G/1. Для практического применения полученных результатов использован известный метод моментов теории вероятностей.

Результаты. Впервые получены спектральные разложения решения интегрального уравнения Линдли для двух систем, с помощью которых выведены расчетные выражения для среднего времени ожидания в очереди в замкнутой форме.

Выводы. Получены спектральные разложения решения интегрального уравнения Линдли для рассматриваемых систем и с их помощью выведены расчетные выражения для среднего времени ожидания в очереди для этих систем в замкнутой форме. Показано, что в системе с запаздыванием во времени среднее время ожидания меньше, чем в обычной системе. Полученные формулы для времени ожидания расширяют и дополняют известную незавершенную формулу теории массового обслуживания для среднего времени ожидания для систем G/G/1 с произвольными законами распределений входного потока и времени обслуживания. Такой подход позволяет рассчитать среднее время ожидания для указанных систем в математических пакетах для широкого диапазона изменения параметров трафика. Все остальные характеристики систем являются производными от времени ожидания.

Кроме среднего времени ожидания, такой подход дает возможность определить и моменты высших порядков времени ожидания. Учитывая тот факт, что вариация задержки пакетов (джиттер) в телекоммуникациях определяется как разброс времени ожидания от его среднего значения, то джиттер можно будет определить через дисперсию времени ожидания.

Полученные результаты публикуется впервые.

КЛЮЧЕВЫЕ СЛОВА: система с запаздыванием, система $E_2/H_2/1$, преобразование Лапласа, среднее время ожидания в очереди, метод спектрального разложения.

ЛІТЕРАТУРА/ЛИТЕРАТУРА

1. Kleinrock L. Queueing Systems / L. Kleinrock // Vol. I: Theory, New York : Wiley, 1975. – 417 p.
2. Brannstrom N. A Queueing Theory analysis of wireless radio systems / N. Brannstrom – Appllied to HS-DSCH. Lulea university of technology, 2004. – 79 p.

DOI 10.15588/1607-3274-2020-3-6

3. Whitt W. Approximating a point process by a renewal process: two basic methods / W. Whitt // Operation Research. – 1982. – № 1. – P. 125–147.
4. Тарасов В. Н. Расширение класса систем массового обслуживания с запаздыванием / В. Н. Тарасов // Автоматика и телемеханика. – 2018. – № 12. – С. 57–70.
5. Тарасов В. Н. Анализ и сравнение двух систем массового обслуживания с гиперэрланговскими входными распределениями / В. Н. Тарасов // Радиоэлектроника, информатика, управление. – 2018. – № 4. – С. 61–70.
6. Тарасов В. Н. Исследование систем массового обслуживания с сдвинутыми эрланговскими и экспоненциальными входными распределениями / В. Н. Тарасов, Н. Ф. Бахарева // Радиоэлектроника, информатика, управление. – 2019. – № 1. – С. 67–76.
7. Тарасов В. Н. Анализ двух систем массового обслуживания HE2/M/1 с обычными и сдвинутыми входными распределениями / В. Н. Тарасов // Радиоэлектроника, информатика, управление. – 2019. – № 2. – С. 71–79.
8. Tarasov V. N. Queueing systems with delay / V. N. Tarasov // Radio Electronics, Computer Science, Control. – 2019. – Vol. 50, № 3. – P. 71–79.
9. Тарасов В. Н. Сравнительный анализ двух систем массового обслуживания M/HE2/1 с обычными и со сдвинутыми входными распределениями / В. Н. Тарасов, Н. Ф. Бахарева // Радиоэлектроника, информатика, управление. – 2019. – № 4. – С. 50–58.
10. Бочаров П. П. Теория массового обслуживания / П. П. Бочаров, А. В. Печинкин. – М. : Изд-во РУДН, 1995. – 529 с.
11. Nonlinear Dynamics in Queueing Theory: Determining the Size of Oscillations in Queues with Delay / [S. Novitzky, J. Pender, R. H. Rand, E. Wesson] // SIAM J. Appl. Dyn. Syst. – 2019. – Vol. 18, № 1. – P. 279–311. DOI: https://doi.org/10.1137/18M1170637.
12. RFC 3393 IP Packet Delay Variation Metric for IP Performance Metrics (IPPM) [Электронный ресурс]. – Режим доступа: https://tools.ietf.org/html/rfc3393.
13. Myskja A. An improved heuristic approximation for the GI/GI/1 queue with bursty arrivals // A. Myskja // Teletraffic and datatraffic in a Period of Change. ITC-13. Elsevier Science Publishers. – 1991. – P. 683–688.
14. Liu X. Diffusion approximations for double-ended queues with reneging in heavy traffic // X. Liu // Queueing Systems: Theory and Applications, Springer. – 2019. – Vol. 91, № 1. – P. 49–87. DOI: 10.1007/s11134-018-9589-7.
15. Poojary S. An asymptotic approximation for TCP CUBIC // S. Poojary, V. Sharma // Queueing Systems: Theory and Applications. – 2019. – Vol. 91, № 1. – P. 171–203. DOI: 10.1007/s11134-018-9594-x.
16. Aras A. K. Many-server Gaussian limits for overloaded non-Markovian queues with customer abandonment / A. K. Aras, X. Chen, Y. Liu // Queueing Systems. – 2018. – Vol. 89, № 1. – P. 81–125. DOI: https://doi.org/10.1007/s11134-018-9575-0.
17. Jennings O. B. Comparisons of ticket and standard queues // O. B. Jennings, J. Pender // Queueing Systems. – 2016. – Vol. 84, No. 1. – P. 145–202. DOI: https://doi.org/10.1007/s11134-016-9493-y.
18. Gromoll H. C. Heavy traffic limit for a tandem queue with identical service times / H. C. Gromoll, B. Terwilliger, B. Zwart // Queueing Systems. – 2018. – Vol. 89, No. 3, P. 213–241. DOI: https://doi.org/10.1007/s11134-017-9560-z.
19. Legros B. M/G/1 queue with event-dependent arrival rates / B. Legros // Queueing Systems. – 2018. – Vol. 89, № 3. – P. 269–301. DOI: https://doi.org/10.1007 /s11134-017-9557-7.
20. Bazhba M. Queue with heavy-tailed Weibull service times / M. Bazhba, J. Blanchet, CH. Rhee // Queueing Systems. – 2019. – Vol. 93, №. 11. – P. 1–32. https://doi.org/10.1007/s11134-019-09640-z.
21. Adan I. Special volume on ‘Recent Developments in Queueing Theory’ of the third ECQT conference / I. Adan, B. D’Auria, O. Kella // Queueing Systems. – 2019. – P. 1–190. DOI: https://doi.org/10.1007/s11134-019-09630-1.
22. Adan I. Special volume on ‘Recent Developments in Queueing Theory’ of the third ECQT conference: part 2 / I. Adan, B. D’Auria, O. Kella // Queueing Systems. – 2019. – P. 1–2. DOI: https://doi.org/10.1007/s11134-019-09637-8.
23. Tibi D. Martingales and buffer overflow for the symmetric shortest queue model / D. Tibi // Queueing Systems. – 2019. – Vol. 93. – P. 153–190. DOI:10.1007/s11134-019-09628-9.
24. Jacobovic R. Asymptotic independence of regenerative processes with a special dependence structure / R. Jacobovic, O. Kella // Queueing Systems. – 2019. – Vol. 93. – P. 139–152. DOI:10.1007/s11134-019-09606-1.

DOI 10.15588/1607-3274-2020-3-6