МАТЕМАТИЧНЕ ТА КОМП'ЮТЕРНЕ МОДЕЛЮВАННЯ

MATHEMATICAL AND COMPUTER MODELING

МАТЕМАТИЧЕСКОЕ И КОМПЬЮТЕРНОЕ МОДЕЛИРОВАНИЕ

UDC 517.988 : 519.632

TWO-SIDED APPROXIMATIONS METHOD BASED ON THE GREEN'S FUNCTIONS USE FOR CONSTRUCTION OF A POSITIVE SOLUTION OF THE DIRICHLE PROBLEM FOR A SEMILINEAR ELLIPTIC EQUATION

Gybkina N. V. – PhD, Associate Professor, Associate Professor of the Department of Applied Mathematics of Kharkiv National University of Radio Electronics, Kharkiv, Ukraine.

Lamtyugova S. M. – PhD, Associate Professor, Associate Professor of the Department of Higher Mathematics of O. M. Beketov National University of Urban Economy in Kharkiv, Kharkiv, Ukraine.

Sidorov M. V. – Dr. Sc., Associate Professor, Professor of the Department of Applied Mathematics of Kharkiv National University of Radio Electronics, Kharkiv, Ukraine.

ABSTRACT

Context. The question of constructing a method of two-sided approximations for finding a positive solution of the Dirichlet problem for a semilinear elliptic equation based on the use of the Green's functions method is considered. The object of research is the first boundary value problem (the Dirichlet problem) for a second-order semilinear elliptic equation.

Objective. The purpose of the research is to develop a method of two-sided approximations for solving the Dirichlet problem for second-order semilinear elliptic equations based on the use of the Green's functions method and to study its work in solving test problems.

Method. Using the Green's functions method, the initial first boundary value problem for a semilinear elliptic equation is replaced by the equivalent Hammerstein integral equation. The integral equation is represented in the form of a nonlinear operator equation with a heterotone operator and is considered in the space of continuous functions, which is semi-ordered using the cone of nonnegative functions. As a solution (generalized) of the boundary value problem, it was taken the solution of the equivalent integral equation. For a heterotone operator, a strongly invariant cone segment is found, the ends of which are the initial approximations for two iteration sequences. The first of these iterative sequences is monotonically increasing and approximates the desired solution to the boundary value problem from below, and the second is monotonically decreasing and approximates it from above. Conditions for the existence of a unique positive solution of the considered Dirichlet problem and two-sided convergence of successive approximations to it are given. General guidelines for constructing a strongly invariant cone segment are also given. The method developed has a simple computational implementation and a posteriori error estimate that is convenient for use in practice.

Results. The method developed was programmed and studied when solving test problems. The results of the computational experiment are illustrated with graphical and tabular informations.

Conclusions. The experiments carried out have confirmed the efficiency and effectiveness of the developed method and make it possible to recommend it for practical use in solving problems of mathematical modeling of nonlinear processes. Prospects for further research may consist the development of two-sided methods for solving problems for systems of partial differential equations, partial differential equations of higher orders and nonstationary multidimensional problems, using semi-discrete methods (for example, the Rothe's method of lines).

KEYWORDS: dirichlet problem for a semilinear elliptic equation, positive solution, strongly invariant conic segment, heterotone operator, method of two-sided approximations, Green's function.

NOMENCLATURE

$C(\bar{\Omega})$ is the Banach space of functions continuous in the domain $\bar{\Omega}$;

$G(\mathbf{x},\mathbf{s})$ is the Green's function of the boundary value problem;

$\mathcal{K}_+$ is a cone of non-negative functions in $C(\bar{\Omega})$;

$K(u_0)$ is a set of functions from $\mathcal{K}_+$ such that $\alpha u_0 \leqslant u \leqslant \beta u_0$, where $\alpha, \beta > 0$;

T is a heterotone operator;

DOI 10.15588/1607-3274-2021-3-3

$\hat{T}$ is a companion operator for the heterotone operator T;

u^* is the exact solution of the boundary value problem;

$\|u\|$ is the norm in the space $C(\bar{\Omega})$;

$< v^0, w^0 >$ is a cone segment, strongly invariant for the heterotone operator T;

$\{v^{(k)}\}$ is a sequence of lower approximations;

v^* is a boundary of the sequence of lower approximations;

$\{w^{(k)}\}$ is a sequence of upper approximations;

w^* is a boundary of the sequence of upper approximations;

Δ is the Laplace operator;

$\kappa > 0$ is a parameter in the Helmholtz operator $\Delta - \kappa^2 u$;

θ is zero element of the Banach space;

$\leqslant$ is a sign of semi-ordering in $C(\bar{\Omega})$, which is introduced by the cone $\mathcal{K}_+$.

INTRODUCTION

The problem of mathematical modeling of many stationary processes considered in chemical kinetics, biology, combustion theory, etc. [1–4], leads to the necessity for finding a positive solution of the Dirichlet problem for a semilinear elliptic equation. Due to this, the problem of developing new and improving existing methods of numerical analysis of this class of problems is relevant.

The object of the study is the Dirichlet problem for a second-order semilinear elliptic equation.

The subject of the research is the method of two-sided approximations for solving the boundary value problems for the second-order semilinear elliptic equations.

Currently there are many methods for the numerical analysis of the boundary value problems for semilinear elliptic equations. Among them, one can single out, in particular, the methods of finite differences, finite elements, boundary integral equations, artificial neural network technique [1, 5–11] or successive approximations with two-sided convergence [12–14]. The methods of the last group allow to construct two sequences of functions that approximate the desired solution of the problem from below and from above, respectively. Due to this, when implementing these methods, one has a convenient a posteriori estimate of the approximation error, and, consequently, a convenient criterion for the termination of iterations. This makes the methods of two-sided approximations more attractive in comparison with other methods that are used to solve boundary value problems for semilinear elliptic equations.

The purpose of the research is to develop a method of two-sided approximations for solving the Dirichlet problem for second-order semilinear elliptic equations based on the use of the Green's functions method and to study its work in solving test problems.

1 PROBLEM STATEMENT

The problem of finding a positive solution of a semilinear elliptic equation with the homogeneous Dirichlet condition is considered in the paper:

$$Lu = f(\mathbf{x}, u), \ \mathbf{x} \in \Omega, \quad (1)$$

$$u(x) > 0, \ \mathbf{x} \in \Omega, \quad (2)$$

$$u|_{\partial\Omega} = 0, \quad (3)$$

where $Lu \equiv -\Delta u$ or $Lu \equiv -\Delta u + \kappa^2 u$, Ω is the Jordan measurable domain from $\mathbb{R}^2$ or $\mathbb{R}^3$ with piecewise smooth boundary $\partial\Omega$ ($\bar{\Omega} = \Omega \cup \partial\Omega$), Δ is the Laplace operator, $\mathbf{x} = (x_1, x_2)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, if $\Omega \subset \mathbb{R}^2$, and $\mathbf{x} = (x_1, x_2, x_3)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, if $\Omega \subset \mathbb{R}^3$.

Let us assume that the function $f(\mathbf{x}, u)$ is continuous and positive for $\mathbf{x} \in \bar{\Omega}$, $u > 0$.

The operator Δ is the Laplace operator and the operator $\Delta - \kappa^2$ is the Helmholtz operator. The problem (1)–(3) often appears as a mathematical model of the nonlinear stationary processes considered in thermal physics, electromagnetism, biology, chemical kinetics, etc. [1, 2, 4]. In this case, the positivity condition (2) naturally arises from the meaning of the function u in a particular field.

2 REVIEW OF THE LITERATURE

The construction of two-sided approximation methods for solving the boundary value problems for partial differential equations is based on the use of the theory of nonlinear operators in semi-ordered spaces.

The theory of linear semi-ordered spaces was constructed by L. V. Kantorovich in the second half of the 30s of the XX century [15, 16]. Further development of the methods of this theory is associated with the works of M. A. Krasnoselsky [17], H. Amann [18], V. I. Opoitsev [19], N. S. Kurpel, B. A. Shuvar [20, 21], A. I. Kolosov [22].

In [17, 18, 23, 24], the existence of the positive solutions of the equations with monotone operators was investigated, and in [19, 25], the solvability of the equations with operators that have the generalized property of monotonicity (the so-called heterotone or mixed monotone operators) was explored. When proving the corresponding theorems of existence, the sequences of functions, which on both sides converged to the solution of the investigated problem, were constructed. As examples of applications of this theory, the questions of the existence of positive solutions of the boundary value problems for nonlinear ordinary differential equations, boundary value

DOI 10.15588/1607-3274-2021-3-3

problems for nonlinear partial differential equations and integral equations were considered. In these works, the theoretical foundations for the development of two-sided iterative schemes were laid, but the iterations themselves were considered by the authors as an auxiliary means of proving the existence theorems for fixed points of operators, and there were no computational results.

In [20, 21], the equations and inequalities in which the operators do not have the monotonicity property are considered, and for them two-sided monotonic iterative processes are constructed. In [22], it was obtained a generalization of the theory of the heterotone operators, which were applied, in particular, to finding the approximate solutions of the boundary value problems with a free boundary for nonlinear ordinary differential equations.

The works [12–14] are devoted to the development of two-sided iterative schemes for solving the boundary value problems for partial differential equations as means of applied mathematics with bringing them to computational implementation. But only problem (1)–(3) with $Lu \equiv -\Delta u$ and $f(\mathbf{x}, u)$ which has a power-law or exponential monotonic nonlinearity was investigated.

The boundary value problems for the equation (1) in case $Lu \equiv -\Delta u + \kappa^2 u$ were not considered.

This work continues the studies begun in [12–14, 26] and is aimed at their generalization and extension to the equation with the Helmholtz operator.

3 MATERIALS AND METHODS

To study the solvability of the problem (1)–(3) and numerically finding its solution, let us construct a method of two-sided approximations using the methods of the theory of nonlinear operators in semi-ordered spaces [17, 19, 27].

If $G(\mathbf{x},\mathbf{s})$ is the Green's function of the problem (1)–(3), then the problem (1)–(3) is equivalent to the Hammerstein's integral equation

$$u(\mathbf{x}) = \int_{\Omega} G(\mathbf{x},\mathbf{s}) f(\mathbf{s},u(\mathbf{s}))d\mathbf{s}. \qquad (4)$$

Let us consider equation (4) in the Banach space $C(\bar{\Omega})$ of functions continuous in $\bar{\Omega}$. The norm in $C(\bar{\Omega})$ is introduced by the rule $\|u\| = \max_{\mathbf{x}\in\bar{\Omega}} |u(\mathbf{x})|$. In $C(\bar{\Omega})$ one selects the cone $\mathcal{K}_+ = \{u \in C(\bar{\Omega}) : u(\mathbf{x}) \geq 0,\ \mathbf{x} \in \bar{\Omega}\}$ of non-negative functions. The cone $\mathcal{K}_+$ in $C(\bar{\Omega})$ is normal (and even sharp). Using the cone $\mathcal{K}_+$ in the space $C(\bar{\Omega})$, let us introduce the semi-ordering by the rule:

$$\text{for } u, v \in C(\bar{\Omega}) \quad u \leqslant v, \text{ if } v - u \in \mathcal{K}_+,$$

that is,

$$u \leqslant v, \text{ if } u(\mathbf{x}) \leq v(\mathbf{x}) \text{ for all } \mathbf{x} \in \bar{\Omega}.$$

If there exists a classical solution of the problem (1)–(3), that is a function $u^* \in C^2(\Omega) \cap C(\bar{\Omega})$ that satisfies the equation (1) and conditions (2), (3), then this function also satisfies the equation (4). If there is no classical solution, then the integral equation (4) can be taken as the basis for the definition of a generalized solution of the problem (1)–(3).

Definition. A solution (generalized) of the boundary value problem (1)–(3) is a function $u^* \in \mathcal{K}_+$, which is a solution of the integral equation (4).

With equation (4) one associates a nonlinear integral operator T acting in $C(\bar{\Omega})$ according to the rule

$$T(u)(\mathbf{x}) = \int_{\Omega} G(\mathbf{x},\mathbf{s}) f(\mathbf{s},u(\mathbf{s}))d\mathbf{s}. \qquad (5)$$

Let us find out what properties the operator T of the form (5) has.

The Green's function $G(\mathbf{x},\mathbf{s})$ is continuous for $\mathbf{x}, \mathbf{s} \in \bar{\Omega}$, $\mathbf{x} \neq \mathbf{s}$, and there are the estimates

$$0 \leq G(\mathbf{x},\mathbf{s}) \leq k_0 \left|\ln \frac{1}{r_{\mathbf{xs}}}\right| \text{ у } \mathbb{R}^2,$$

$$0 \leq G(\mathbf{x},\mathbf{s}) \leq \frac{k_0}{r_{\mathbf{xs}}} \text{ у } \mathbb{R}^3,$$

where $r_{\mathbf{xs}} = |\mathbf{x} - \mathbf{s}|$ is the distance between points $\mathbf{x}$ and $\mathbf{s}$.

From these estimates and from the conditions imposed on the function $f(\mathbf{x}, u)$, it follows that the operator T of the form (5) acts in the space $C(\bar{\Omega})$ and leaves the cone $\mathcal{K}_+$ invariant, that is, T is a positive operator.

Also, the operator T of the form (5) is u_0-positive operator, where the function $u_0(\mathbf{x})$ belongs to $\mathcal{K}_+ \setminus \{\theta\}$, is defined by the equality

$$u_0(\mathbf{x}) = \int_{\Omega} G(\mathbf{x},\mathbf{s})d\mathbf{s} \qquad (6)$$

and is the solution of the problem

$$Lu = 1,\ \mathbf{x} \in \Omega,$$
$$u\big|_{\partial\Omega} = 0.$$

The property of u_0-positivity follows from the fact [17]: if Ω_0 is some subdomain of the domain Ω, moreover, $\mu(\Omega_0) > 0$, then there is such $\gamma = \gamma(\Omega_0) > 0$, that the following inequality holds

$$\gamma \int_{\Omega} G(\mathbf{x},\mathbf{s})d\mathbf{s} \leq \int_{\Omega_0} G(\mathbf{x},\mathbf{s})d\mathbf{s}.$$

DOI 10.15588/1607-3274-2021-3-3

If $u \in \mathcal{K}_+ \setminus \{\theta\}$, then for some $\alpha_0 > 0$ there is a set $\Omega_0 \subset \Omega$ such that $\mu(\Omega_0) > 0$ and $f(\mathbf{x}, u(\mathbf{x})) \geq \alpha_0$ for all $\mathbf{x} \in \Omega_0$. Then for all $\mathbf{x} \in \bar{\Omega}$

$$T(u)(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{s}) f(\mathbf{s},u(\mathbf{s}))d\mathbf{s} \geq$$
$$\geq \int_{\Omega_0} G(\mathbf{x},\mathbf{s}) f(\mathbf{s},u(\mathbf{s}))d\mathbf{s} \geq$$
$$\geq \alpha_0 \int_{\Omega_0} G(\mathbf{x},\mathbf{s})d\mathbf{s} \geq \alpha_0\gamma \int_\Omega G(\mathbf{x},\mathbf{s})d\mathbf{s} = \alpha_0\gamma u_0(\mathbf{x}) .$$

On the other hand, for all $\mathbf{x} \in \bar{\Omega}$

$$T(u)(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{s}) f(\mathbf{s},u(\mathbf{s}))d\mathbf{s} \leq$$
$$\leq \max_{\mathbf{x}\in\bar{\Omega}} f(\mathbf{x},u(\mathbf{x})) \cdot \int_\Omega G(\mathbf{x},\mathbf{s})d\mathbf{s} =$$
$$= \max_{\mathbf{x}\in\bar{\Omega}} f(\mathbf{x},u(\mathbf{x})) \cdot u_0(\mathbf{x}) .$$

Thus, there will be double inequality for all $\mathbf{x} \in \bar{\Omega}$

$$\alpha u_0(\mathbf{x}) \leq \int_\Omega G(\mathbf{x},\mathbf{s}) f(\mathbf{s},u(\mathbf{s}))d\mathbf{s} \leq \beta u_0(\mathbf{x}) , \tag{7}$$

where $\alpha = \alpha_0\gamma > 0$, $\beta = \max_{\mathbf{x}\in\bar{\Omega}} f(\mathbf{x},u(\mathbf{x})) > 0$, which is the definition of the u_0-positivity of the operator T.

A constructive study of the equation (4) (and, consequently, of the problem (1)–(3)) with the opportunity of constructing two-sided approximations to its positive solution is possible if the function $f(\mathbf{x},u)$ has the monotonicity property.

Let the function $f(\mathbf{x},u)$ allows a diagonal representation $f(\mathbf{x},u) = \hat{f}(\mathbf{x},u,u)$, where the nonnegative function $\hat{f}(\mathbf{x},v,w)$, continuous in the set of variables $\mathbf{x}$, v, w, monotonically increases with respect to v and monotonically decreases with respect to w for all $\mathbf{x} \in \Omega$. Then the operator T of the form (5) will be heterotone one with the companion operator

$$\hat{T}(v, w)(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{s}) \hat{f}(\mathbf{s}, v(\mathbf{s}), w(\mathbf{s}))d\mathbf{s} . \tag{8}$$

Obviously, the operators T and $\hat{T}$ are completely continuous.

If the function $f(\mathbf{x},u)$ monotonically increases with respect to u for all $\mathbf{x} \in \Omega$, then one can choose $\hat{f}(\mathbf{x},v,w) = f(\mathbf{x},v)$ and the companion operator is defined by the equality

$$\hat{T}(v, w)(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{s}) f(\mathbf{s}, v(\mathbf{s}))d\mathbf{s} . \tag{9}$$

For a function $f(\mathbf{x},u)$ monotonically decreasing with respect to u, one can put $\hat{f}(\mathbf{x},v,w) = f(\mathbf{x},w)$, and then the companion operator will have the form

$$\hat{T}(v, w)(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{s}) f(\mathbf{s}, w(\mathbf{s}))d\mathbf{s} . \tag{10}$$

If for any positive numbers v, w, for any $\tau \in (0, 1)$

$$\hat{f}\left(\mathbf{x}, \tau v, \frac{1}{\tau} w\right) > \tau \hat{f}(\mathbf{x}, v, w) ,\ \mathbf{x} \in \Omega , \tag{11}$$

then the heterotone operator T of the form (5) for which the operator $\hat{T}$ of the form (8) is companion one will be pseudo-concave and, moreover, u_0-pseudo-concave with the function $u_0(\mathbf{x})$ of the form (6).

Actually, for any $v, w \in \mathcal{K}_+ \setminus \{\theta\}$, the inequality (7) implies that

$$\alpha u_0(\mathbf{x}) \leq \int_\Omega G(\mathbf{x}, \mathbf{s}) \hat{f}(\mathbf{s}, v(\mathbf{s}), w(\mathbf{s}))d\mathbf{s} \leq \beta u_0(\mathbf{x}) , \tag{12}$$

where $\alpha > 0$, $\beta > 0$, that is $\hat{T}(v, w) \in K(u_0)$ for any $v, w \in \mathcal{K}_+ \setminus \{\theta\}$. Here, $K(u_0)$ is a set of functions from $\mathcal{K}_+$ such that $\alpha u_0 \leqslant u \leqslant \beta u_0$, where $\alpha, \beta > 0$.

Suppose now that $v, w \in K(u_0)$, thus there are $\alpha_1 = \alpha_1(v) > 0$, $\beta_1 = \beta_1(v) > 0$, $\alpha_2 = \alpha_2(w) > 0$, $\beta_2 = \beta_2(w) > 0$ such that $\alpha_1 u_0 \leqslant v \leqslant \beta_1 u_0$, $\alpha_2 u_0 \leqslant w \leqslant \beta_2 u_0$. For $\tau \in (0, 1)$ consider the difference $\hat{T}\left(\tau v, \frac{1}{\tau} w\right) - \tau \hat{T}(v, w)$:

$$\hat{T}\left(\tau v, \frac{1}{\tau} w\right)(\mathbf{x}) - \tau \hat{T}(v, w)(\mathbf{x}) =$$
$$= \int_\Omega G(\mathbf{x}, \mathbf{s})\left[\hat{f}\left(\mathbf{s}, \tau v(\mathbf{s}), \frac{1}{\tau} w(\mathbf{s})\right) - \tau \hat{f}(\mathbf{s}, v(\mathbf{s}), w(\mathbf{s}))\right] d\mathbf{s} .$$

Then it follows from the inequality (11) and the condition of continuity of the function $\hat{f}(\mathbf{x},v,w)$ that $\hat{T}\left(\tau v, \frac{1}{\tau} w\right) - \tau \hat{T}(v, w) \geqslant \theta$, while $\hat{T}\left(\tau v, \frac{1}{\tau} w\right) - \tau \hat{T}(v, w) \neq \theta$, which means the pseudo-concavity of the operator T.

Further, from the inequalities (11), (12) it follows that for all $\mathbf{x} \in \bar{\Omega}$

$$\hat{T}\left(\tau v, \frac{1}{\tau} w\right)(\mathbf{x}) - \tau \hat{T}(v, w)(\mathbf{x}) =$$
$$= \int_\Omega G(\mathbf{x}, \mathbf{s})\left[\hat{f}\left(\mathbf{s}, \tau v(\mathbf{s}), \frac{1}{\tau} w(\mathbf{s})\right) - \tau \hat{f}(\mathbf{s}, v(\mathbf{s}), w(\mathbf{s}))\right] ds \geq$$
$$\geq \alpha_1 u_0(\mathbf{x}) ,$$

DOI 10.15588/1607-3274-2021-3-3

where $\alpha_1 > 0$.

Then, applying inequality (12) again, one gets that for all $\mathbf{x} \in \bar{\Omega}$

$$\int\limits_{\Omega} G(\mathbf{x},\mathbf{s})\, \hat{f}\left(\mathbf{s}, \tau v(\mathbf{s}), \frac{1}{\tau} w(\mathbf{s})\right) d\mathbf{s} \geq$$
$$\geq \tau \int\limits_{\Omega} G(\mathbf{x},\mathbf{s})\, \hat{f}(\mathbf{s}, v(\mathbf{s}), w(\mathbf{s})) d\mathbf{s} + \alpha_1 u_0(\mathbf{x}) \geq$$
$$\geq \tau\left(1 + \frac{\alpha_1}{\beta\tau}\right) \int\limits_{\Omega} G(\mathbf{x},\mathbf{s})\, \hat{f}(\mathbf{s}, v(\mathbf{s}), w(\mathbf{s})) d\mathbf{s} .$$

So for all $\mathbf{x} \in \bar{\Omega}$ there is the inequality

$$\int\limits_{\Omega} G(\mathbf{x},\mathbf{s})\, \hat{f}\left(\mathbf{s}, \tau v(\mathbf{s}), \frac{1}{\tau} w(\mathbf{s})\right) d\mathbf{s} \geq$$
$$\geq \tau(1+\eta) \int\limits_{\Omega} G(\mathbf{x},\mathbf{s})\, \hat{f}(\mathbf{s}, v(\mathbf{s}), w(\mathbf{s})) d\mathbf{s} ,$$

where $\eta = \eta(v,\, w,\, \tau) = \dfrac{\alpha_1}{\beta\tau} > 0$, which means u_0-pseudo-concavity of the operator T.

Thus, the following statement holds.

Lemma. The operator T of the form (5), where $G(\mathbf{x},\mathbf{s})$ is the Green's function of the problem (1)–(3), considered in the space $C(\bar{\Omega})$, semi-ordered by the cone $\mathcal{K}_+$ of nonnegative functions, has the following properties:

a) it is a positive operator;

b) it is u_0-positive operator, where the function $u_0(\mathbf{x})$ is defined by the equation (6);

c) it is a heterotone operator for which the operator $\hat{T}$ of the form (8) is a companion one if the function $f(\mathbf{x},u)$ allows a diagonal representation $f(\mathbf{x},u) = \hat{f}(\mathbf{x},u,u)$, where the function $\hat{f}(\mathbf{x},v,w)$, continuous in the set of variables $\mathbf{x}$, v, w, monotonically increases with respect to v and monotonically decreases with respect to w for all $\mathbf{x} \in \Omega$;

d) if inequality (11) holds, then it is a pseudo-concave and even u_0-pseudo-concave operator, where the function $u_0(\mathbf{x})$ has the form (6).

Further one will assume that the operator T of the form (5) is heterotone one with the companion operator of the form (8). Let us construct a method of two-sided approximations for finding a positive solution of the integral equation (4) (and, therefore, of the boundary value problem (1)–(3)).

In the cone $\mathcal{K}_+$ one distinguishes a strongly invariant cone segment $< v^0, w^0 >$ by the conditions $\hat{T}(v^0, w^0) \geqslant v^0$, $\hat{T}(w^0, v^0) \leqslant w^0$, which for the operator $\hat{T}$, that is determined by the equality (8), take the form: for all $\mathbf{x} \in \bar{\Omega}$

$$\int\limits_{\Omega} G(\mathbf{x},\, \mathbf{s})\, \hat{f}(\mathbf{s},\, v^0(\mathbf{s}),\, w^0(\mathbf{s})) d\mathbf{s} \geq v^0(\mathbf{x}) , \qquad (13)$$

$$\int\limits_{\Omega} G(\mathbf{x},\, \mathbf{s})\, \hat{f}(\mathbf{s},\, w^0(\mathbf{s}),\, v^0(\mathbf{s})) d\mathbf{s} \leq w^0(\mathbf{x}) . \qquad (14)$$

According to the scheme

$$v^{(k+1)} = \hat{T}(v^{(k)}, w^{(k)}) ,$$
$$w^{(k+1)} = \hat{T}(w^{(k)}, v^{(k)})$$

one will form an iterative process

$$v^{(k+1)}(\mathbf{x}) = \int\limits_{\Omega} G(\mathbf{x},\, \mathbf{s})\, \hat{f}(\mathbf{s},\, v^{(k)}(\mathbf{s}),\, w^{(k)}(\mathbf{s})) d\mathbf{s} , \qquad (15)$$

$$w^{(k+1)}(\mathbf{x}) = \int\limits_{\Omega} G(\mathbf{x},\, \mathbf{s})\, \hat{f}(\mathbf{s},\, w^{(k)}(\mathbf{s}),\, v^{(k)}(\mathbf{s})) d\mathbf{s} , \qquad (16)$$

$$k = 0,\, 1,\, 2,\, \ldots ,$$

$$v^{(0)}(\mathbf{x}) = v^0(\mathbf{x}) ,\ w^{(0)}(\mathbf{x}) = w^0(\mathbf{x}) . \qquad (17)$$

Due to the strong invariance of the cone segment $< v^0, w^0 >$ and the heterotonicity of the operator T, for which the operator $\hat{T}$ is a companion one, let us conclude that the sequence $\{v^{(k)}(\mathbf{x})\}$ does not decrease along the cone $\mathcal{K}_+$, and the sequence $\{w^{(k)}(\mathbf{x})\}$ does not increase along the cone $\mathcal{K}_+$. In addition, from the normality of the cone $\mathcal{K}_+$ and the complete continuity of the operator $\hat{T}$ it follows that there are boundaries $v^*(\mathbf{x})$ and $w^*(\mathbf{x})$ of these sequences. So, the chain of inequalities is confirmed

$$v^0 = v^{(0)} \leqslant v^{(1)} \leqslant \ldots \leqslant v^{(k)} \leqslant \ldots \leqslant v^* \leqslant$$
$$\leqslant w^* \leqslant \ldots \leqslant w^{(k)} \leqslant \ldots \leqslant w^{(1)} \leqslant w^{(0)} = w^0 .$$

There are two possible cases: $v^* < w^*$ and $v^* = w^*$. In the second case $u^* := v^* = w^*$ is the only fixed point of the operator T on the cone segment $< v^0, w^0 >$, and therefore u^* is the unique solution of the considered boundary value problem on $< v^0, w^0 >$.

Functions $v^*(\mathbf{x})$ and $w^*(\mathbf{x})$ is the solution of the system of equations $v = \hat{T}(v,w)$, $w = \hat{T}(w,v)$, which in this case has the form:

$$v(\mathbf{x}) = \int\limits_{\Omega} G(\mathbf{x},\mathbf{s})\, \hat{f}(\mathbf{s}, v(\mathbf{s}), w(\mathbf{s})) d\mathbf{s} , \qquad (18)$$

$$w(\mathbf{x}) = \int\limits_{\Omega} G(\mathbf{x},\mathbf{s})\, \hat{f}(\mathbf{s}, w(\mathbf{s}), v(\mathbf{s})) d\mathbf{s} . \qquad (19)$$

DOI 10.15588/1607-3274-2021-3-3

The equality $v^* = w^*$ will be satisfied if the system (18), (19) does not have on $< v^0, w^0 >$ such solutions that $v \neq w$.

Thus, the following theorem holds.

Theorem 1. Let $< v^0, w^0 >$ be a strongly invariant cone segment for a heterotone operator T of the form (5) with the companion operator $\hat{T}$ of the form (8) and the system of equations (18), (19) has no solutions on $< v^0, w^0 >$ such that $v \neq w$. Then the iterative process (15)–(17) converges in the norm of the space $C(\bar{\Omega})$ to the unique on $< v^0, w^0 >$ continuous positive solution u^* of the boundary value problem (1)–(3), and a chain of inequalities holds

$$v^0 = v^{(0)} \leqslant v^{(1)} \leqslant \ldots \leqslant v^{(k)} \leqslant \ldots \leqslant u^* \leqslant$$
$$\leqslant \ldots \leqslant w^{(k)} \leqslant \ldots \leqslant w^{(1)} \leqslant w^{(0)} = w^0 . \quad (20)$$

As one sees, it follows from the chain of inequalities (20) that each of the cone segments $< v^{(k)}, w^{(k)} >$, $k = 0, 1, 2, \ldots$, is a strongly invariant cone segment for the heterotone operator T of the form (5) with the companion operator $\hat{T}$ of the form (8).

The chain of inequalities (20) characterizes the iterative process (15)–(17) as two-sided one.

The conditions for the existence of a unique positive solution of the boundary value problem (1)–(3) and the two-sided convergence of successive approximations (15)–(17) to it can be refined by clarifying the conditions under which the system of equations (18), (19) does not have solutions such that $v \neq w$ on some of the strongly invariant cone segments $< v^{(k)}, w^{(k)} >$, $k = 0, 1, 2, \ldots$.

One of the conditions that will ensure the implementation of the equality $v^* = w^*$ is the condition of the existence of such $\gamma \in (0;1)$ that

$$\left\| \hat{T}(v,w) - \hat{T}(w,v) \right\| \leq \gamma \left\| v - w \right\|$$

for all $v, w \in < v^0, w^0 >$.

Suppose there exists such number $L > 0$ that the function $\hat{f}(\mathbf{x}, v, w)$ for all numbers v, w such that $0 < v,\, w < M_0$, where $M_0 = \max_{\mathbf{x} \in \bar{\Omega}} w^0(\mathbf{x})$, and for all $\mathbf{x} \in \Omega$ satisfies the inequality

$$\left| \hat{f}(\mathbf{x}, w, v) - \hat{f}(\mathbf{x}, v, w) \right| \leq L \left| w - v \right| . \quad (21)$$

Then

$$\left\| \hat{T}(w,v) - \hat{T}(v,w) \right\| = \max_{\mathbf{x} \in \bar{\Omega}} [\hat{T}(w,v)(\mathbf{x}) - \hat{T}(v,w)(\mathbf{x})] =$$
$$= \max_{\mathbf{x} \in \bar{\Omega}} \int_{\Omega} G(\mathbf{x}, \mathbf{s}) [\hat{f}(\mathbf{s}, w(\mathbf{s}), v(\mathbf{s})) - \hat{f}(\mathbf{s}, v(\mathbf{s}), w(\mathbf{s}))] d\mathbf{s} \leq$$
$$\leq LM \max_{\mathbf{x} \in \bar{\Omega}} [w(\mathbf{x}) - v(\mathbf{x})] = LM \left\| w - v \right\| ,$$

where $M = \max_{\mathbf{x} \in \bar{\Omega}} u_0(\mathbf{x})$.

Therefore,

$$\left\| \hat{T}(w,v) - \hat{T}(v,w) \right\| \leq LM \left\| w - v \right\| . \quad (22)$$

It is clear that inequality (22) implies an estimate

$$\left\| w^{(k+1)} - v^{(k+1)} \right\| \leq (LM)^{k+1} \left\| w^{(0)} - v^{(0)} \right\| . \quad (23)$$

Thus, the equality $v^* = w^*$ will hold if $\gamma = LM < 1$, and then the following theorem is valid.

Theorem 2. Let $< v^0, w^0 >$ be a strongly invariant cone segment for the heterotone operator T of the form (5) with the companion operator $\hat{T}$ of the form (8) and the condition (21) holds, moreover, $\gamma = LM < 1$, where $M = \max_{\mathbf{x} \in \bar{\Omega}} u_0(\mathbf{x})$. Then the iterative process (15)–(17) converges bilaterally in the norm of the space $C(\bar{\Omega})$ to a unique on $< v^0, w^0 >$ continuous positive solution u^* of the boundary value problem (1)–(3), and the estimate (23) takes place.

Another condition that the system of equations (18), (19) does not have solutions such that $v \neq w$ on a strongly invariant cone segment $< v^0, w^0 >$ is the condition that the heterotone operator T of the form (5) with the companion operator $\hat{T}$ of the form (8) is u_0-pseudo-concave. Then, taking into account assertion d) of the lemma, one arrives at the following result.

Theorem 3. Let $< v^0, w^0 > \subset K(u_0)$ be a strongly invariant conic segment for the heterotone operator T of the form (5) with the companion operator $\hat{T}$ of the form (8) and the condition (11) holds. Then the iterative process (15)–(17) converges bilaterally in the norm of the space $C(\bar{\Omega})$ to a unique on $< v^0, w^0 >$ continuous positive solution u^* of the boundary value problem (1)–(3).

Let us now consider the partial cases when the function $f(\mathbf{x}, u)$ only monotonically increases or only monotonically decreases with respect to u.

If the function $f(\mathbf{x}, u)$ monotonically increases with respect to u and $\hat{f}(\mathbf{x}, v, w) = f(\mathbf{x}, v)$ is chosen, then the companion operator $\hat{T}$ is given by the equality (9), and the conditions (13), (14), which distinguish a strongly

DOI 10.15588/1607-3274-2021-3-3

invariant cone segment $< v^0, w^0 >$ (in this case, the strong invariance coincides with the common invariance of the operator T), look like: for all $\mathbf{x} \in \bar{\Omega}$

$$\int_\Omega G(\mathbf{x},\mathbf{s})\, f(\mathbf{s}, v^0(\mathbf{s}))d\mathbf{s} \ge v^0(\mathbf{x}) , \qquad (24)$$

$$\int_\Omega G(\mathbf{x},\mathbf{s})\, f(\mathbf{s}, w^0(\mathbf{s}))d\mathbf{s} \le w^0(\mathbf{x}) . \qquad (25)$$

As one can see, each of the inequalities (24), (25) independently of the other ones distinguishes its end of the cone segment $< v^0, w^0 >$.

For the function $f(\mathbf{x},u)$ monotonic in u, the system of equations (18), (19) has the form

$$v(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{s})\, f(\mathbf{s}, v(\mathbf{s}))d\mathbf{s} ,$$

$$w(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{s})\, f(\mathbf{s}, w(\mathbf{s}))d\mathbf{s}$$

and the condition that this system does not have on $< v^0, w^0 >$ solutions such that $v \ne w$ turns into the condition for the existence of a unique solution of the equation (4).

The condition (21) turns into the usual Lipschitz condition for the function $f(\mathbf{x},u)$: there exists a number $L > 0$ such that the function $f(\mathbf{x},u)$ for all numbers v, w such that $0 < v,\ w < M_0$, where $M_0 = \max\limits_{\mathbf{x}\in\bar{\Omega}} w^0(\mathbf{x})$, and for all $\mathbf{x} \in \Omega$ satisfies the inequality

$$\left| f(\mathbf{x}, v) - f(\mathbf{x}, w)\right| \le L\left|v - w\right| , \qquad (26)$$

and condition (11) of u_0-pseudo-concavity takes the following form: for any positive number u and for any $\tau \in (0, 1)$

$$f(\mathbf{x},\ \tau u) > \tau f(\mathbf{x},\ u) ,\ \mathbf{x} \in \Omega . \qquad (27)$$

Thus, if there exists a strongly invariant cone segment $< v^0, w^0 >$, distinguished by the conditions (24), (25), and at least one of the conditions (26) or (27) is satisfied, then successive approximations that are formed according to the scheme

$$v^{(k+1)}(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{s})\, f(\mathbf{s}, v^{(k)}(\mathbf{s}))d\mathbf{s} ,\ k = 0, 1, 2, \ldots , \qquad (28)$$

$$w^{(k+1)}(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{s})\, f(\mathbf{s}, w^{(k)}(\mathbf{s}))d\mathbf{s} ,\ k = 0, 1, 2, \ldots , \qquad (29)$$

$$v^{(0)}(\mathbf{x}) = v^0(\mathbf{x}) ,\ w^{(0)}(\mathbf{x}) = w^0(\mathbf{x}) , \qquad (30)$$

converge bilaterally to a unique on $< v^0, w^0 >$ continuous positive solution u^* of the boundary value problem (1)–(3).

As can be seen from (28)–(30), for a function $f(\mathbf{x},u)$ monotonic with respect to u, the lower $\{v^{(k)}(\mathbf{x})\}$ and upper $\{w^{(k)}(\mathbf{x})\}$ approximations form two independent sequences, and in the computational organization of the iterative process, their formation can be carried out using the technologies of computations parallelization.

For a function $f(\mathbf{x},u)$, which decreases monotonically with respect to u, if $\hat{f}(\mathbf{x},v,w) = f(\mathbf{x},w)$ is chosen, the companion operator $\hat{T}$ is given by the equality (10), and the conditions (13), (14), which distinguish a strongly invariant cone segment $< v^0, w^0 >$, take the form: for all $\mathbf{x} \in \bar{\Omega}$

$$\int_\Omega G(\mathbf{x},\mathbf{s})\, f(\mathbf{s}, w^0(\mathbf{s}))d\mathbf{s} \ge v^0(\mathbf{x}) , \qquad (31)$$

$$\int_\Omega G(\mathbf{x},\mathbf{s})\, f(\mathbf{s}, v^0(\mathbf{s}))d\mathbf{s} \le w^0(\mathbf{x}) . \qquad (32)$$

The system of the equations (18), (19) for the antitone with respect to u function $f(\mathbf{x},u)$ has the form

$$v(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{s})\, f(\mathbf{s}, w(\mathbf{s}))d\mathbf{s} ,$$

$$w(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{s})\, f(\mathbf{s}, v(\mathbf{s}))d\mathbf{s} .$$

The condition (21), as in the monotone case, turns into the usual Lipschitz condition for the function $f(\mathbf{x},u)$: there exists a number $L > 0$ such that the function $f(\mathbf{x},u)$ for all numbers v, w such that $0 < v,\ w < M_0$, where $M_0 = \max\limits_{\mathbf{x}\in\bar{\Omega}} w^0(\mathbf{x})$, and for all $\mathbf{x} \in \Omega$ satisfies the inequality

$$\left| f(\mathbf{x}, w) - f(\mathbf{x}, v)\right| \le L\left|v - w\right| , \qquad (33)$$

and the condition (11) of u_0-pseudo-concavity takes the following form: for any positive number u and for any $\tau \in (0, 1)$

$$f\left(\mathbf{x},\ \frac{1}{\tau}u\right) > \tau f(\mathbf{x},\ u) ,\ \mathbf{x} \in \Omega . \qquad (34)$$

Thus, if there exists a strongly invariant cone segment $< v^0, w^0 >$, distinguished by the conditions (31), (32), and at least one of the conditions (33) or (34) is satisfied,

DOI 10.15588/1607-3274-2021-3-3

then successive approximations that are formed according to the scheme

$$v^{(k+1)}(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \mathbf{s}) f(\mathbf{s}, w^{(k)}(\mathbf{s}))d\mathbf{s}, \ k = 0, 1, 2, \ldots,$$

$$w^{(k+1)}(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \mathbf{s}) f(\mathbf{s}, v^{(k)}(\mathbf{s}))d\mathbf{s}, \ k = 0, 1, 2, \ldots,$$

$$v^{(0)}(\mathbf{x}) = v^0(\mathbf{x}), \ w^{(0)}(\mathbf{x}) = w^0(\mathbf{x}),$$

converge bilaterally to a unique on $< v^0, w^0 >$ continuous positive solution u^* of the boundary value problem (1)–(3).

At the k-th iteration, as an approximate solution of the boundary value problem (1)–(3), the following function is taken

$$u^{(k)}(\mathbf{x}) = \frac{w^{(k)}(\mathbf{x}) + v^{(k)}(\mathbf{x})}{2}. \quad (35)$$

Then one will have a convenient a posteriori error estimate for the approximate solution (35):

$$\left\| u^* - u^{(k)} \right\| \le \frac{1}{2} \max_{\mathbf{x} \in \bar{\Omega}} (w^{(k)}(\mathbf{x}) - v^{(k)}(\mathbf{x})),$$

which is an undoubted advantage of the constructed two-way iterative process.

So, if the accuracy $\varepsilon > 0$ is given, then the iterative process should be carried out until the inequality

$$\max_{\mathbf{x} \in \bar{\Omega}} (w^{(k)}(\mathbf{x}) - v^{(k)}(\mathbf{x})) < 2\varepsilon \quad (36)$$

is satisfied and with an accuracy of ε it can be assumed that

$$u^*(\mathbf{x}) \approx u^{(k)}(\mathbf{x}).$$

In addition, from the conditions of Theorem 3 an a priori estimate of the error can be written:

$$\left\| u^* - u^{(k)} \right\| \le \frac{\gamma^k}{2} \max_{\mathbf{x} \in \bar{\Omega}} (w^0(\mathbf{x}) - v^0(\mathbf{x})).$$

Then from the inequality

$$\left\| u^* - u^{(k)} \right\| \le \frac{\gamma^k}{2} \max_{\mathbf{x} \in \bar{\Omega}} (w^0(\mathbf{x}) - v^0(\mathbf{x})) < \varepsilon$$

one finds that to achieve the accuracy of ε it is necessary to make

$$k_0(\varepsilon) = \left[\frac{\ln \dfrac{\max_{\mathbf{x} \in \bar{\Omega}} (w^0(\mathbf{x}) - v^0(\mathbf{x}))}{2\varepsilon}}{\ln \dfrac{1}{LM}} \right] + 1$$

iterations, where square brackets denote an integer part of the number.

The strongly invariant cone segment $< v^0, w^0 >$, which is distinguished by the conditions (13), (14), is an a priori estimate for the unknown exact solution u^*. We further it will be given the general recommendations for finding a segment $< v^0, w^0 >$. Since $\hat{T}(v, w) \in K(u_0)$ for any $v, w \in \mathcal{K}_+ \setminus \{\theta\}$, the ends of a strongly invariant cone segment $< v^0, w^0 >$ can be sought in the form $v^0(\mathbf{x}) = \alpha u_0(\mathbf{x})$, $w^0(\mathbf{x}) = \beta u_0(\mathbf{x})$, where $0 < \alpha < \beta$. Then the inequalities (14), (15) take the form: for all $\mathbf{x} \in \bar{\Omega}$

$$\int_{\Omega} G(\mathbf{x}, \mathbf{s}) \hat{f}(\mathbf{s}, \alpha u_0(\mathbf{s}), \beta u_0(\mathbf{s}))d\mathbf{s} \ge \alpha u_0(\mathbf{x}),$$

$$\int_{\Omega} G(\mathbf{x}, \mathbf{s}) \hat{f}(\mathbf{s}, \beta u_0(\mathbf{s}), \alpha u_0(\mathbf{s}))d\mathbf{s} \le \beta u_0(\mathbf{x}).$$

These inequalities can be reduced to a form

$$\alpha \le \min_{\mathbf{x} \in \bar{\Omega}} h_1(\mathbf{x}; \alpha, \beta), \ \beta \ge \max_{\mathbf{x} \in \bar{\Omega}} h_2(\mathbf{x}; \alpha, \beta), \quad (37)$$

where

$$h_1(\mathbf{x}; \alpha, \beta) = \int_{\Omega} \frac{G(\mathbf{x}, \mathbf{s})}{u_0(\mathbf{x})} \hat{f}(\mathbf{s}, \alpha u_0(\mathbf{s}), \beta u_0(\mathbf{s}))d\mathbf{s},$$

$$h_2(\mathbf{x}; \alpha, \beta) = \int_{\Omega} \frac{G(\mathbf{x}, \mathbf{s})}{u_0(\mathbf{x})} \hat{f}(\mathbf{s}, \beta u_0(\mathbf{s}), \alpha u_0(\mathbf{s}))d\mathbf{s}.$$

The value of $\max_{\mathbf{x} \in \bar{\Omega}} (w^0(\mathbf{x}) - v^0(\mathbf{x})) = (\beta - \alpha)M$ should be as small as possible for faster convergence of iterations, and therefore, in the practical implementation of the iterative process (15)–(17) one should take the largest α and the smallest β, satisfying inequalities (37).

4 EXPERIMENTS

The computational experiment was performed for the problems (1)–(3) with the right part of the form

$$f(\mathbf{x}, u) = \lambda u^p + \mu u^{-q}, \quad (38)$$

where $p, q > 0$, $\lambda, \mu > 0$.

DOI 10.15588/1607-3274-2021-3-3

For a function $f(\mathbf{x},u)$ of the form (38) it was chosen $\hat{f}(\mathbf{x},v,w)=\lambda v^p+\mu w^{-q}$ and the corresponding operators (5), (8) have the form

$$T(u)(\mathbf{x})=\int_\Omega G(\mathbf{x},\mathbf{s})[\lambda u(\mathbf{s})^p+\mu u(\mathbf{s})^{-q}]d\mathbf{s},$$

$$\hat{T}(v,w)(\mathbf{x})=\int_\Omega G(\mathbf{x},\mathbf{s})[\lambda v(\mathbf{s})^p+\mu w(\mathbf{s})^{-q}]d\mathbf{s},$$

where $G(\mathbf{x},\mathbf{s})$ is the Green's function for the operator $Lu\equiv-\Delta u$ or $Lu\equiv-\Delta u+\kappa^2 u$ in the area Ω.

The pseudo-concavity condition (11), written for the function $f(\mathbf{x},u)$ of the form (38), leads to the inequality

$$\lambda\tau(\tau^{p-1}-1)v^p+\mu\tau(\tau^{q-1}-1)w^{-q}>0,$$

which will be performed for all $\tau\in(0,1)$, $v,w>0$, if $0<p<1$, $0<q<1$.

The ends of a strongly invariant cone segment $<v^0,w^0>$ will be sought in the form $v^0(\mathbf{x})=\alpha u_0(\mathbf{x})$, $w^0(\mathbf{x})=\beta u_0(\mathbf{x})$, where $0<\alpha<\beta$, and the function $u_0(\mathbf{x})$ has the form (6). Further, let us find

$$h_1(\mathbf{x};\alpha,\beta)=$$
$$=\int_\Omega \frac{G(\mathbf{x},\mathbf{s})}{u_0(\mathbf{x})}\{\lambda[\alpha u_0(\mathbf{s})]^p+\mu[\beta u_0(\mathbf{s})]^{-q}\}d\mathbf{s}=$$
$$=\lambda\alpha^p\Psi(\mathbf{x})+\frac{\mu}{\beta^q}\Theta(\mathbf{x}),$$
$$h_2(\mathbf{x};\alpha,\beta)=$$
$$=\int_\Omega \frac{G(\mathbf{x},\mathbf{s})}{u_0(\mathbf{x})}\{\lambda[\beta u_0(\mathbf{s})]^p+\mu[\alpha u_0(\mathbf{s})]^{-q}\}d\mathbf{s}=$$
$$=\lambda\beta^p\Psi(\mathbf{x})+\frac{\mu}{\alpha^q}\Theta(\mathbf{x}),$$

where

$$\Psi(\mathbf{x})=\int_\Omega \frac{G(\mathbf{x},\mathbf{s})}{u_0(\mathbf{x})}[u_0(\mathbf{s})]^p d\mathbf{s},$$
$$\Theta(\mathbf{x})=\int_\Omega \frac{G(\mathbf{x},\mathbf{s})}{u_0(\mathbf{x})}[u_0(\mathbf{s})]^{-q} d\mathbf{s}.$$

Let us denote

$$m_1=\min_{\mathbf{x}\in\bar\Omega}\Psi(\mathbf{x}),\ M_1=\max_{\mathbf{x}\in\bar\Omega}\Psi(\mathbf{x}),$$
$$m_2=\min_{\mathbf{x}\in\bar\Omega}\Theta(\mathbf{x}),\ M_2=\max_{\mathbf{x}\in\bar\Omega}\Theta(\mathbf{x}).$$

Then inequalities (36) for finding α, β, take the form

$$\alpha\le\lambda m_1\alpha^p+\mu m_2\beta^{-q},\quad \beta\ge\lambda M_1\beta^p+\mu M_2\alpha^{-q}. \quad (39)$$

The iterative process (15)–(17) in this case will take the form

$$v^{(k+1)}(\mathbf{x})=\int_\Omega G(\mathbf{x},\mathbf{s})\left\{\lambda[v^{(k)}(\mathbf{s})]^p+\frac{\mu}{[w^{(k)}(\mathbf{s})]^q}\right\}d\mathbf{s}, \quad (40)$$

$$w^{(k+1)}(\mathbf{x})=\int_\Omega G(\mathbf{x},\mathbf{s})\left\{\lambda[w^{(k)}(\mathbf{s})]^p+\frac{\mu}{[v^{(k)}(\mathbf{s})]^q}\right\}d\mathbf{s}, \quad (41)$$

$$k=0,1,2,\ldots,$$

$$v^{(0)}(\mathbf{x})=\alpha u_0(\mathbf{x}),\ w^{(0)}(\mathbf{x})=\beta u_0(\mathbf{x}). \quad (42)$$

Thus, if $0<p<1$, $0<q<1$, then the problem (1)–(3) with $Lu\equiv-\Delta u$ or $Lu\equiv-\Delta u+\kappa^2 u$ and a function $f(\mathbf{x},u)$, which has the form (38), for any $\lambda,\mu>0$ has a unique positive solution $u^*(\mathbf{x})$, to which the iterative process (40)–(42) converges bilaterally.

In order to construct an approximate solution of the problem (1)–(3) it is necessary to find α, β ($0<\alpha<\beta$) as a solution of the system of inequalities (39) (the highest value α and the lowest value β will be the solution of the corresponding system) and taking some $\varepsilon>0$ (calculation accuracy), implement the iterative process (40)–(42) to perform the inequality (36). The approximate solution of the problem will be further determined by the formula (35).

The numerical implementation of the process (40)–(42) was performed using the PYTHON language. For computational experiments it was chosen $\varepsilon=10^{-4}$, the integrals in (40), (41) were calculated with the accuracy 10^{-6} by an adaptive procedure based on Gaussian quadrature with previous piecewise linear interpolation with the same accuracy of functions $v^{(k)}(\mathbf{x})$, $w^{(k)}(\mathbf{x})$.

5 RESULTS

The results of solving the problem (1)–(3) in a unit circle $\Omega=\{\mathbf{x}=(x_1,x_2): x_1^2+x_2^2<1\}$ and a unit sphere $\Omega=\{\mathbf{x}=(x_1,x_2,x_3): x_1^2+x_2^2+x_3^2<1\}$ with $Lu\equiv-\Delta u$ and $Lu\equiv-\Delta u+\kappa^2 u$ for the function $f(\mathbf{x},u)$ of the form (38) for $p=q=\frac{1}{2}$ and $\lambda=\mu=1$ are given in Tables 1–4 and in Fig. 1–10.

Let us consider the results of a computational experiment for the problem (1)–(3) with the Laplace operator: $Lu\equiv-\Delta u$. In the case of a unit circle, the Green's function of the Laplace operator has the form

DOI 10.15588/1607-3274-2021-3-3

$$G(r,\ \varphi,\ \rho,\ \psi)=\frac{1}{4\pi}\ln\frac{r^2\rho^2-2r\rho\cos(\varphi-\psi)+1}{r^2-2r\rho\cos(\varphi-\psi)+\rho^2},$$

$$x_1=r\cos\varphi,\ x_2=r\sin\varphi,\ s_1=\rho\cos\psi,\ s_2=\rho\sin\psi,$$

$$0\le r<1,\ 0\le\varphi<2\pi,$$

and in the case of a unit sphere it has the form

$$G(r,\ \theta,\ \varphi,\ \rho,\ \vartheta,\ \psi)=$$

$$=\frac{1}{4\pi}\left[\frac{1}{\sqrt{r^2-2r\rho\cos\gamma+\rho^2}}-\frac{1}{\sqrt{r^2\rho^2-2r\rho\cos\gamma+1}}\right],$$

$$\cos\gamma=\cos\theta\cos\vartheta+\sin\theta\sin\vartheta\cos(\varphi-\psi),$$

$$x_1=r\sin\theta\cos\varphi,\ x_2=r\sin\theta\sin\varphi,\ x_3=r\cos\theta,$$

$$s_1=\rho\sin\vartheta\cos\psi,\ s_2=\rho\sin\vartheta\sin\psi,\ s_3=\rho\cos\vartheta,$$

$$0\le r<1,\ 0\le\theta\le\pi,\ 0\le\varphi<2\pi.$$

For the problem (1)–(3) with the right-hand side of the form (38) and the Laplace operator considered in a unit circle, it is found that $\alpha=0.83292$, $\beta=1.97354$, and the accuracy $\varepsilon=10^{-4}$ was achieved on the 12-th iteration, and $\left\|u^{(12)}\right\|=0.5636$.

Figure 1 shows graphs of the upper $w^{(k)}(\mathbf{x})$ (solid line) and lower $v^{(k)}(\mathbf{x})$ approximations (dashed line), $k=0,1,...,12$, in cross section $x_2=0$.

Table 1 shows the values of the approximate solution $u^{(12)}(\mathbf{x})$ of the problem (1)–(3) with the Laplace operator considered in a unit circle at the points $(x_1^{(i)},\ 0)=(0{,}1i;\ 0)$, $i=0,1,...,10$.

Each iteration was performed for an average of 185 sec., the total operating time of the program was 37 minutes.

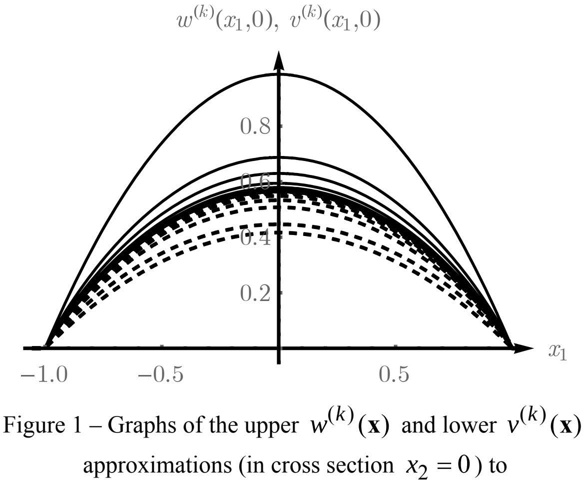

Figure 1 – Graphs of the upper $w^{(k)}(\mathbf{x})$ and lower $v^{(k)}(\mathbf{x})$ approximations (in cross section $x_2=0$) to the solution of problem (1)–(3) with the Laplace operator considered in a unit circle

Table 1 – Values of the approximate solution $u^{(12)}(\mathbf{x})$ of the problem (1)–(3) with the Laplace operator considered in a unit circle at the points $(x_1^{(i)},\ 0)=(0{,}1i;\ 0)$, $i=0,1,...,10$

$(x_1^{(i)},0)$	(0, 0)	(0.1; 0)	(0.2; 0)	(0.3; 0)	(0.4; 0)
$u^{(12)}(x_1^{(i)},0)$	0.5636	0.5584	0.5427	0.5166	0.4798
$(x_1^{(i)},0)$	(0.5; 0)	(0.6; 0)	(0.7; 0)	(0.8; 0)	(0.9; 0)
$u^{(12)}(x_1^{(i)},0)$	0.4322	0.3734	0.3028	0.2195	0.1211
$(x_1^{(i)},0)$	(1; 0)				
$u^{(12)}(x_1^{(i)},0)$	0				

Figures 2 and 3 show the surface and the level lines of the approximate solution $u^{(12)}(\mathbf{x})$, respectively.

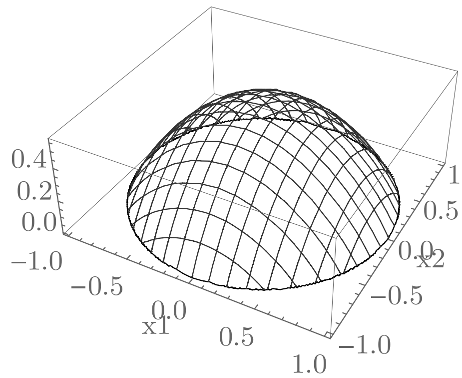

Figure 2 – Graph of the surface of the approximate solution $u^{(12)}(\mathbf{x})$ of the problem (1)–(3) with the Laplace operator considered in a unit circle

For the problem (1)–(3) with the right-hand side of the form (38) and the Laplace operator considered in a unit sphere it is found that $\alpha=0.56568$, $\beta=1.73173$, and the accuracy $\varepsilon=10^{-4}$ was achieved on the 12-th iteration, and $\left\|u^{(12)}\right\|=0.4134$.

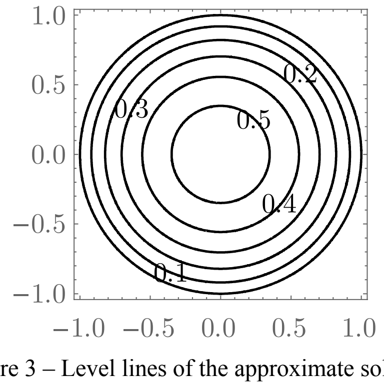

Figure 3 – Level lines of the approximate solution $u^{(12)}(\mathbf{x})$ of the problem (1)–(3) with the Laplace operator considered in a unit circle

DOI 10.15588/1607-3274-2021-3-3

Figure 4 shows graphs of the upper $w^{(k)}(\mathbf{x})$ (solid line) and lower $v^{(k)}(\mathbf{x})$ approximations (dashed line), $k = 0, 1, ..., 12$, in cross section $x_2 = 0$, $x_3 = 0$.

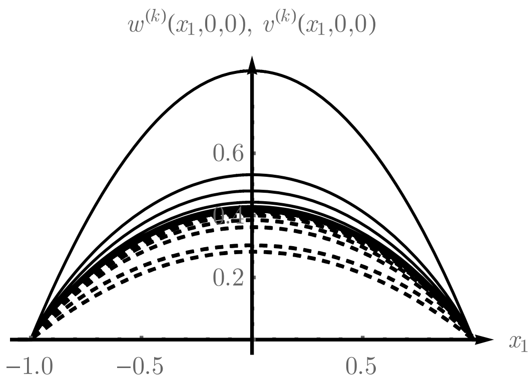

Figure 4 – Graphs of the upper $w^{(k)}(\mathbf{x})$ and lower $v^{(k)}(\mathbf{x})$ approximations (in cross section $x_2 = 0$, $x_3 = 0$) to the solution of problem (1)–(3) with the Laplace operator considered in a unit sphere

Table 2 shows the values of the approximate solution $u^{(12)}(\mathbf{x})$ of the problem (1)–(3) with the Laplace operator considered in a unit sphere at the points $(x_1^{(i)}, 0) = (0{,}1i; 0)$, $i = 0, 1, ..., 10$.

Each iteration was performed for an average of 207 sec., the total operating time of the program was 41,4 minutes.

Figure 5 shows, respectively, the level surfaces of the approximate solution $u^{(12)}(\mathbf{x})$.

Let us now consider the results of a computational experiment for the problem (1)–(3) with the Helmholtz operator: $Lu \equiv -\Delta u + \kappa^2 u$. In the case of a unit circle, the Green's function of the Helmholtz operator at $\kappa = 1$ has the form

$$G(r, \varphi, \rho, \psi) =$$
$$= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{w_{nm}^{(1)}(r,\varphi)w_{nm}^{(1)}(\rho,\psi) + w_{nm}^{(2)}(r,\varphi)w_{nm}^{(2)}(\rho,\psi)}{\|w_{nm}\|^2(\lambda_{nm}+1)},$$

Table 2 – Values of the approximate solution $u^{(12)}(\mathbf{x})$ of the problem (1)–(3) with the Laplace operator considered in a unit sphere at the points $(x_1^{(i)}, 0) = (0{,}1i; 0)$, $i = 0, 1, ..., 10$

$(x_1^{(i)},0,0)$	(0, 0, 0)	(0,1; 0, 0)	(0.2; 0; 0)	(0.3; 0; 0)
$u^{(12)}(x_1^{(i)},0,0)$	0.4134	0.4098	0.3987	0.3803
$(x_1^{(i)},0)$	(0.4; 0; 0)	(0.5; 0; 0)	(0.6; 0; 0)	(0.7; 0; 0)
$u^{(12)}(x_1^{(i)},0)$	0.3542	0.3204	0.2783	0.2275
$(x_1^{(i)},0)$	(0.8; 0; 0)	(0.9; 0; 0)	(1; 0; 0)	
$u^{(12)}(x_1^{(i)},0)$	0.1666	0.0936	0	

DOI 10.15588/1607-3274-2021-3-3

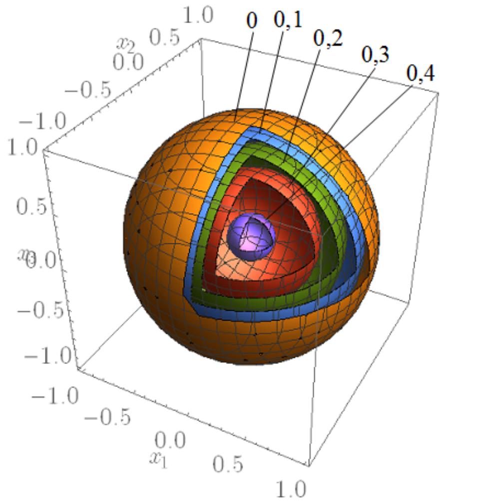

Figure 5 – Level surfaces of the approximate solution $u^{(12)}(\mathbf{x})$ of the problem (1)–(3) with the Laplace operator considered in a unit sphere

$\lambda_{nm} = \mu_{nm}^2$, μ_{nm} is m-th positive root of the equation $J_n(\mu) = 0$,

$$w_{nm}^{(1)}(r,\varphi) = J_n(r\sqrt{\lambda_{nm}})\cos n\varphi,$$
$$w_{nm}^{(2)}(r,\varphi) = J_n(r\sqrt{\lambda_{nm}})\sin n\varphi,$$
$$w_{nm}^{(1)}(\rho,\psi) = J_n(\rho\sqrt{\lambda_{nm}})\cos n\psi,$$
$$w_{nm}^{(2)}(\rho,\psi) = J_n(\rho\sqrt{\lambda_{nm}})\sin n\psi,$$
$$\|w_{nm}\|^2 = \frac{\pi}{2}(1+\delta_{n0})[J_n'(\mu_{nm})]^2,$$
$$\delta_{n0} = \begin{cases} 1, & n = 0, \\ 0, & n = 1, 2, ..., \end{cases}$$

$x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $s_1 = \rho\cos\psi$, $s_2 = \rho\sin\psi$, $0 \le r < 1$, $0 \le \varphi < 2\pi$,

and in the case of a unit sphere it has the form

$$G(r, \theta, \varphi, \rho, \vartheta, \psi) =$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \frac{w_{nmk}^{(1)}(r,\theta,\varphi)w_{nmk}^{(1)}(\rho,\vartheta,\psi)}{\|w_{nmk}\|^2(\lambda_{nk}+1)} +$$
$$+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{w_{nmk}^{(2)}(r,\theta,\varphi)w_{nmk}^{(2)}(\rho,\vartheta,\psi)}{\|w_{nmk}\|^2(\lambda_{nk}+1)},$$

$\lambda_{nk} = \mu_{nk}^2$, μ_{nk} is k-th positive root of the equation $J_{n+\frac{1}{2}}(\mu) = 0$,

$$w_{nmk}^{(1)}(r,\theta,\varphi) = \frac{1}{\sqrt{r}} J_{n+\frac{1}{2}}(\mu_{nk} r) P_n^m(\cos\theta)\cos m\varphi,$$
$$w_{nmk}^{(2)}(r,\theta,\varphi) = \frac{1}{\sqrt{r}} J_{n+\frac{1}{2}}(\mu_{nk} r) P_n^m(\cos\theta)\sin m\varphi,$$

$$w_{nmk}^{(1)}(\rho,\vartheta,\psi)=\frac{1}{\sqrt{\rho}}J_{n+\frac{1}{2}}(\mu_{nk}\rho)P_n^m(\cos\vartheta)\cos m\psi\ ,$$

$$w_{nmk}^{(2)}(\rho,\vartheta,\psi)=\frac{1}{\sqrt{\rho}}J_{n+\frac{1}{2}}(\mu_{nk}\rho)P_n^m(\cos\vartheta)\sin m\psi\ ,$$

$P_n^m(z)$ are associated Legendre functions,

$$\left\|w_{nmk}^{(1)}\right\|^2=\left\|w_{nmk}^{(2)}\right\|^2=$$

$$=\frac{\pi R^2(1+\delta_{m0})(n+m)!}{(2n+1)(n-m)!}\left[J'_{n+\frac{1}{2}}(\mu_{nk})\right]^2,$$

$$\delta_{ij}=\begin{cases}1, & i=j,\\ 0, & i\neq j,\end{cases}$$

$$x_1=r\sin\theta\cos\varphi\ ,\ x_2=r\sin\theta\sin\varphi\ ,\ x_3=r\cos\theta\ ,$$

$$s_1=\rho\sin\vartheta\cos\psi\ ,\ s_2=\rho\sin\vartheta\sin\psi\ ,\ s_3=\rho\cos\vartheta\ ,$$

$$0\le r<1\ ,\ 0\le\theta\le\pi\ ,\ 0\le\varphi<2\pi\ .$$

For the problem (1)–(3) with the right-hand side of the form (38) and the Helmholtz operator at $\kappa=1$ considered in a unit circle it is found that $\alpha=0.69148$, $\beta=1.93356$, and the accuracy $\varepsilon=10^{-4}$ was achieved on the 12-th iteration, and $\left\|u^{(12)}\right\|=0.4853$.

Figure 6 shows graphs of the upper $w^{(k)}(\mathbf{x})$ (solid line) and lower $v^{(k)}(\mathbf{x})$ approximations (dashed line), $k=0, 1, ..., 12$, in cross section $x_2=0$.

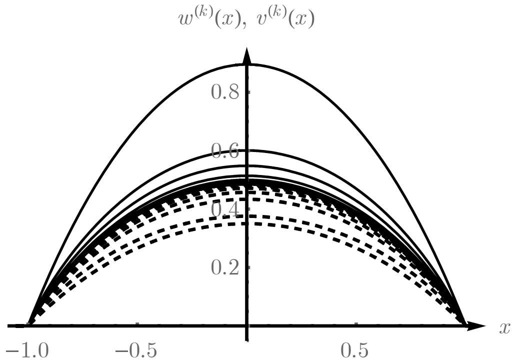

Figure 6 – Graphs of the upper $w^{(k)}(\mathbf{x})$ and lower $v^{(k)}(\mathbf{x})$ approximations (in cross section $x_2=0$) to the solution of problem (1)–(3) with the Helmholtz operator considered in a unit circle

Table 3 shows the values of the approximate solution $u^{(12)}(\mathbf{x})$ of the problem (1)–(3) with the Helmholtz operator considered in a unit circle at the points $(x_1^{(i)}, 0)=(0,1i; 0)$, $i=0, 1, ..., 10$.

Table 3 – Values of the approximate solution $u^{(12)}(\mathbf{x})$ of the problem (1)–(3) with the Helmholtz operator considered in a unit circle at the points $(x_1^{(i)}, 0)=(0,1i; 0)$, $i=0, 1, ..., 10$

$(x_1^{(i)},0)$	(0, 0)	(0.1; 0)	(0.2; 0)	(0.3; 0)	(0,4; 0)
$u^{(12)}(x_1^{(i)},0)$	0.4853	0.4811	0.4687	0.4478	0.4182
$(x_1^{(i)},0)$	(0.5; 0)	(0.6; 0)	(0.7; 0)	(0.8; 0)	(0.9; 0)
$u^{(12)}(x_1^{(i)},0)$	0.3793	0.3305	0.2709	0.1988	0.1114
$(x_1^{(i)},0)$	(1; 0)				
$u^{(12)}(x_1^{(i)},0)$	0				

Each iteration was performed for an average of 195 sec., the total operating time of the program was 39 minutes.

Figures 7 and 8 show the surface and the level lines of the approximate solution $u^{(12)}(\mathbf{x})$, respectively.

For the problem (1)–(3) with the right-hand side of the form (38) and the Helmholtz operator at $\kappa=1$ considered in a unit sphere it is found that $\alpha=0,49891$, $\beta=1,74257$, and the accuracy $\varepsilon=10^{-4}$ was achieved on the 12-th iteration, and $\left\|u^{(12)}\right\|=0,3761$.

Figure 9 shows graphs of the upper $w^{(k)}(\mathbf{x})$ (solid line) and lower $v^{(k)}(\mathbf{x})$ approximations (dashed line), $k=0, 1, ..., 12$, in cross section $x_2=0$, $x_3=0$.

Table 4 shows the values of the approximate solution $u^{(12)}(\mathbf{x})$ of the problem (1)–(3) with the Helmholtz operator considered in a unit sphere at the points $(x_1^{(i)}, 0)=(0,1i; 0)$, $i=0, 1, ..., 10$.

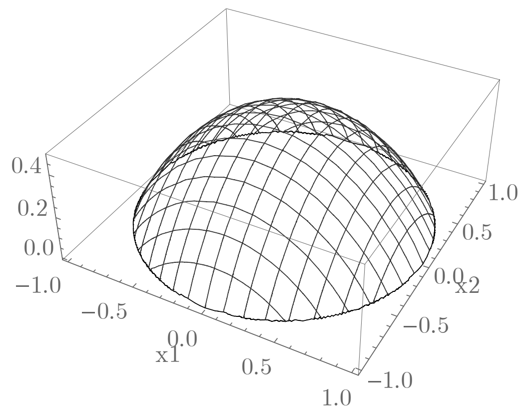

Figure 7 – Graph of the surface of the approximate solution $u^{(12)}(\mathbf{x})$ of the problem (1)–(3) with the Helmholtz operator considered in a unit circle

DOI 10.15588/1607-3274-2021-3-3

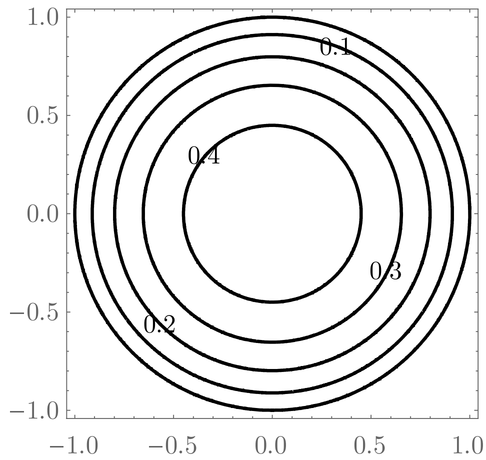

Figure 8 – Level lines of the approximate solution $u^{(12)}(\mathbf{x})$ of the problem (1)–(3) with the Helmholtz operator considered in a unit circle

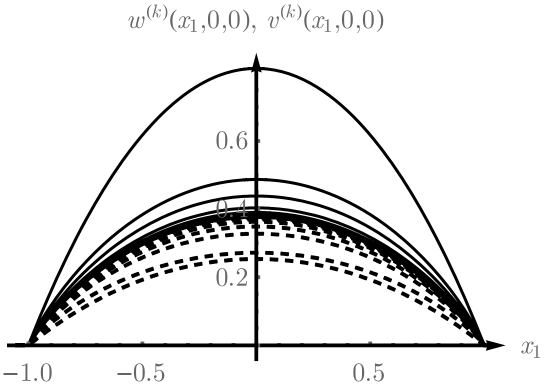

Figure 9 – Graphs of the upper $w^{(k)}(\mathbf{x})$ and lower $v^{(k)}(\mathbf{x})$ approximations (in cross section $x_2 = 0$, $x_3 = 0$) to the solution of problem (1)–(3) with the Helmholtz operator considered in a unit sphere

Table 4 – Values of the approximate solution $u^{(12)}(\mathbf{x})$ of the problem (1)–(3) with the Helmholtz operator considered in a unit sphere at the points $(x_1^{(i)}, 0) = (0,1i;\ 0)$, $i = 0, 1, ..., 10$

$(x_1^{(i)},0,0)$	(0, 0, 0)	(0.1; 0, 0)	(0.2; 0; 0)	(0.3; 0; 0)
$u^{(12)}(x_1^{(i)},0,0)$	0.3761	0.3729	0.3635	0.3477
$(x_1^{(i)},0)$	(0.4; 0; 0)	(0.5; 0; 0)	(0.6; 0; 0)	(0.7; 0; 0)
$u^{(12)}(x_1^{(i)},0)$	0.3253	0.2957	0.2586	0.2130
$(x_1^{(i)},0)$	(0.8; 0; 0)	(0.9; 0; 0)	(1; 0; 0)	
$u^{(12)}(x_1^{(i)},0)$	0.1575	0.0895	0	

Each iteration was performed for an average of 223 sec., the total operating time of the program was 44,6 minutes.

Figure 10 shows, respectively, the level surfaces of the approximate solution $u^{(12)}(\mathbf{x})$.

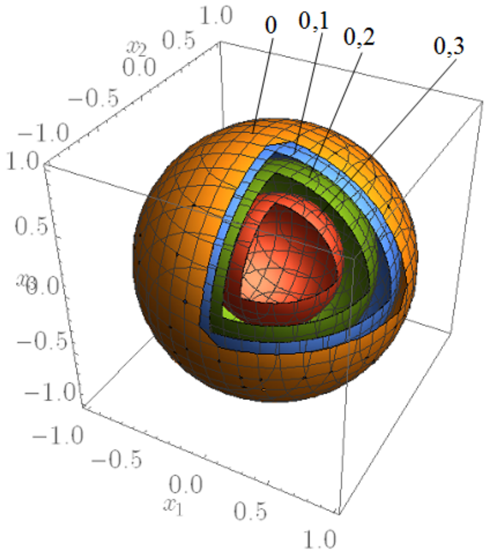

Figure 10 – Level surfaces of the approximate solution $u^{(12)}(\mathbf{x})$ of the problem (1)–(3) with the Helmholtz operator considered in a unit sphere

6 DISCUSSION

The analysis of the results shows that the method of two-sided approximations is an effective numerical method for solving the boundary value problems of the form (1)–(3) with the Laplace and Helmholtz operators. Its advantages include convenient a posteriori error estimation, iteration completion criterion and easy to implement algorithm. Analyzing the results of the computational experiment, one can see that for the test problem in the approximate solution, the correct sign after the comma is set in about two or three iterations. Considering the relation $\frac{\varepsilon_{k+1}}{\varepsilon_k}$, $k = 0, 1, ..., 10$, where $\varepsilon_k = \frac{1}{2}\max_{\mathbf{x}\in\bar{\Omega}}(w^{(k)}(\mathbf{x}) - v^{(k)}(\mathbf{x}))$, k is the number of iteration, it can be seen that the iterative sequence has a geometric rate of convergence. At the same time, when switching to a three-dimensional problem, the program runtime increased, but the convergence rate remained almost unchanged. It can also be noted that in the transition to the case of three-dimensional space, the length of the initial strongly invariant cone segment decreased, which indicates a better (compared to the two-dimensional case) choice of initial approximations. The solution norm in the three-dimensional problem also turned out to be less than for the two-dimensional one.

Figures 1, 4, 6, and 9 clearly demonstrate the two-sided nature of the convergence of the constructed iterative sequences $\{v^{(k)}(\mathbf{x})\}$ and $\{w^{(k)}(\mathbf{x})\}$ according to the chain of inequalities (20): at each k-th iteration the unknown exact solution $u^*(\mathbf{x})$ of the problem is above the approximation $v^{(k)}(\mathbf{x})$ and below the approximation $w^{(k)}(\mathbf{x})$.

DOI 10.15588/1607-3274-2021-3-3

CONCLUSIONS

The problem of constructing two-sided approximations to the positive solution of the first boundary value problem for a semilinear elliptic equation of the second order with the Laplace operator and the Helmholtz operator is solved.

The **scientific novelty** of the obtained results is that the method of two-sided approximations for solving nonlinear operator equations with a heterotone operator in terms of its application to the boundary value problems for a semilinear elliptic equation of the second order has been further developed, and for equations with the Helmholtz operator this method is used for the first time. The developed method has a number of advantages, such as a convenient a posteriori estimation of the error of the approximate solution and a simple computational algorithm. This distinguishes it from other numerical methods for solving nonlinear boundary value problems for second-order semilinear elliptic equations and makes it attractive for application in engineering practice.

The **practical significance** of the results obtained lies in the fact that the method proposed has shown itself well in solving test problems, allows fast software implementation, which will permit carrying out highly invariant computational experiments when solving practical problems of mathematical modeling of nonlinear processes.

The limitations in using the method can be associated with the conditions imposed on the behavior of the function $f(\mathbf{x},u)$, which are stated in Theorems 1, 2, and 3, and also with the fact that for the area Ω the Green's function of the corresponding differential operator must be known.

The **prospects** for further research are the extension of the method of two-sided approximations developed in this work to the boundary value problems for equations of elliptic type with other boundary conditions and to initial boundary value problems for parabolic and hyperbolic equations, using semi-discrete methods (for example, the Rothe's method of lines).

ACKNOWLEDGMENTS

The work was performed at the Department of Applied Mathematics of the Kharkiv National University of Radio Electronics and the Department of Higher Mathematics of the O. M. Beketov National University of Urban Economy in Kharkiv within the framework of scientific research conducted by the departments.

REFERENCES

1. Pao C. V. Nonlinear parabolic and elliptic equations. New York, Plenum Press, 1992, 794 p. DOI: 10.1007/978-1-4615-3034-3
2. Samarskii A. A., Samarskii A. P. Principles of Mathematical Modelling: Ideas, Methods, Examples. London, CRC Press, 2001, 360 p. DOI: https://doi.org/10.1201/9781482288131
3. Kolmanovskii V., Myshkis A. Models, In: *Applied Theory of Functional Differential Equations. Mathematics and Its Applications (Soviet Series).* Dordrecht, Springer, 1992, Vol. 85, pp. 1–34. DOI: 10.1007/978-94-015-8084-7_1
4. Frank-Kamenetskii D. A. Diffusion and Heat Exchange in Chemical Kinetics. Princeton, Princeton University Press, 1955, 384 p. DOI: 10.1515/9781400877195

DOI 10.15588/1607-3274-2021-3-3

5. Samarskii A. A. The Theory of Difference Schemes. New York, CRC Press, 2001, 786 p. DOI: 10.1201/9780203908518
6. Chen G., Zhou J., Ni W.-M. Algorithms and visualization for solutions of nonlinear elliptic equations, *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, 2000, Vol. 10, No. 7, pp. 1565–1612. DOI: 10.1142/S0218127400001006
7. Kumar M., Kumar P. A finite element approach for finding positive solutions of semilinear elliptic Dirichlet problems, *Numerical Methods for Partial Differential Equations: An International Journal*, 2009, Vol. 25, No. 5, pp. 1119–1128. DOI: 10.1002/num.20390
8. Kumar M., Mishra G. A review on nonlinear elliptic partial differential equations and approaches for solution, *International Journal of Nonlinear Science*, 2012, Vol. 13, No. 4, pp. 401–418. DOI: IJNS.2012.06.30/621
9. Kumar M., Mishra G. An introduction to numerical methods for the solutions of partial differential equations, *Applied Mathematics*, 2011, Vol. 2, No. 11, pp. 1327–1338. DOI: 10.4236/am.2011.211186
10. Matinfar M., Nemati K. A numerical extension on a convex nonlinear elliptic problem, *International Mathematical Forum,* 2008, Vol. 3, No. 17, pp. 811–816.
11. Yadav N., Yadav A., Deep K. Artificial neural network technique for solution of nonlinear elliptic boundary value problems, *Proceedings of Fourth International Conference on Soft Computing for Problem Solving*. New Delhi, Springer, 2015, pp. 113–121. DOI: 10.1007/978-81-322-2217-0_10
12. Kolosova S. V., Lukhanin V. S., Sidorov M. V. On positive solutions of Liouville-Gelfand problem, *Vestnik Kazahskogo nacional'nogo universiteta. Serija matematika, mehanika, informatika*, 2018, No. 3 (99), pp. 78–91.
13. Konchakovska O., Sidorov M. Numerical analysis of the one-dimensional nonlinear boundary value problem that modeling an electrostatic NEMS by two-sided approximations method, *Journal of Numerical Analysis, Industrial and Applied Mathematics (JNAIAM),* 2020, Vol. 14, No. 3–4. – pp. 17–26.
14. Sidorov M. V. Green-Rvachev's quasi-function method for constructing two-sided approximations to positive solution of nonlinear boundary value problems, *Carpathian Mathematical Publications*, 2018, Vol. 10, No. 2, pp. 360–375. DOI: 10.15330/cmp.10.2.360-375
15. Kantorovich L. V., Akilov G. P. Functional Analysis. New York, Pergamon, 1982, 600 p. DOI: 10.1016/C2013-0-03044-7
16. Kantorovitch L. Linear operations in semi-ordered spaces. I, *Matematicheskij sbornik*, 1940. Vol. 7 (49), No. 2, pp. 209–284.
17. Krasnosel'skii M. A. Positive Solutions of Operator Equations. Groningen, Noordhoff, 1964, 381 p.
18. Amann H. Fixed Point Equations and Nonlinear Eigenvalue Problems in Ordered Banach Spaces, *SIAM Review*, 1976, Vol. 18, No. 4, pp. 620–709. DOI: 10.1137/1018114
19. Opojtsev V. I. A Generalization of the Theory of Monotone and Concave Operators, *Transactions of the Moscow Mathematical Society*, 1979, Vol. 36, pp. 243–279.
20. Kurpel' N. S., Grechko V. I. On Some Modifications of Chaplygin's Method for Equations in Partially Ordered Spaces, *Ukrainian Mathematical Journal*, 1973, Vol. 25, pp. 30–36. DOI: 10.1007/BF01085389
21. Shuvar B. A., Kopach M. I. Two-sided Operator Inequalities with Nonmonotone Operators, *Differential Equations,* 2006. Vol. 42, pp. 586–590. DOI: 10.1134/S001226610604015X
22. Kolosov A. I. A Boundary Value Problem on a Nonfixed Interval, *Siberian Mathematical Journal,* 1976, Vol. 17, No. 6, pp. 944–948. DOI: https://doi.org/10.1007/BF00968020
23. Felipe-Sosa R., Felipe R. Boundary value problems for nonlinear elliptic equations with two parameters, *Differential Equations and Dynamical Systems*, 2014, Vol. 22, No. 1, pp. 1–13. DOI: 10.1007/s12591-013-0187-1

24. Franca M. Positive solutions of semilinear elliptic equations: a dynamical approach, *Differential and Integral Equations,* 2013, Vol. 26, No. 5/6, pp. 505–554.
25. Rus M.-D. The method of monotone iterations for mixed monotone operators : Ph.D. Thesis Summary. Cluj-Napoca, 2010, 45 p.
26. Sidorov M. V. Method of Two-Sided Approximations of the Solution of the First Boundary Value Problem for Nonlinear Ordinary Differential Equations Based on the Green's Function Use, *Radio Electronics, Computer Science, Control.* – 2019, No. 1 (48), pp. 57–66. DOI: 10.15588/1607-3274-2019-1-6
27. Vulikh B. Z. Introduction to the Theory of Partially Ordered Spaces. Groningen, Wolters-Noordhoff, 1967, 403 p.

Received 28.04.2021.
Accepted 30.07.2021.

УДК 517.988 : 519.632

МЕТОД ДВОБІЧНИХ НАБЛИЖЕНЬ НА ОСНОВІ ВИКОРИСТАННЯ ФУНКЦІЇ ГРІНА ПОБУДОВИ ДОДАТНОГО РОЗВ'ЯЗКУ ЗАДАЧІ ДІРІХЛЕ ДЛЯ НАПІВЛІНІЙНОГО ЕЛІПТИЧНОГО РІВНЯННЯ

Гибкіна Н. В. – канд. техн. наук, доцент, доцент каф. прикладної математики, Харківський національний університет радіоелектроніки, Харків, Україна.

Ламтюгова С. М. – канд. фіз.-мат. наук, доцент, доцент каф. вищої математики, Харківський національний університет міського господарства імені О. М. Бекетова, Харків, Україна.

Сидоров М. В. – д-р фіз.-мат. наук, доцент, професор каф. прикладної математики, Харківский національній університет радіоелектроніки, Харків, Україна.

АНОТАЦІЯ

Актуальність. Розглянуто питання побудови методу двобічних наближень знаходження додатного розв'язку задачі Діріхле для напівлінійного еліптичного рівняння на основі використання метода функцій Гріна. Об'єктом дослідження є перша крайова задача (задача Діріхле) для напівлінійного еліптичного рівняння другого порядку.

Мета. Метою роботи є розробка на основі використання методу функцій Гріна методу двобічних наближень розв'язання задачі Діріхле для напівлінійних еліптичних рівнянь другого порядку і дослідження його роботи при розв'язанні тестових задач.

Метод. За допомогою методу функцій Гріна вихідна перша крайова задача для напівлінійного еліптичного рівняння замінюється еквівалентним інтегральним рівнянням Гаммерштейна. Інтегральне рівняння подається у вигляді нелінійного операторного рівняння з гетеротонним оператором і розглядається у просторі неперервних функцій, який напівупорядковано за допомогою конуса невід'ємних функцій. За розв'язок (узагальнений) крайової задачі приймаємо розв'язок еквівалентного інтегрального рівняння. Для гетеротонного оператора знаходиться сильно інваріантний конусний відрізок, кінці якого є початковими наближеннями для двох ітераційних послідовностей. Перша з цих ітераційних послідовностей є монотонно зростаючою і наближає шуканий розв'язок крайової задачі знизу, а друга є монотонно спадною і наближає його зверху. Наведено умови існування єдиного додатного розв'язку розглядуваної задачі Діріхле та двобічної збіжності до нього послідовних наближень. Також наведено загальні рекомендації з побудови сильно інваріантного конусного відрізка. Розроблений метод має просту обчислювальну реалізацію і зручну для використання на практиці апостеріорну оцінку похибки.

Результати. Розроблений метод програмно реалізовано та досліджено при розв'язанні тестових задач. Результати обчислювального експерименту проілюстровано графічною та табличною інформаціями.

Висновки. Проведені експерименти підтвердили працездатність та ефективність розробленого метода і дозволяють рекомендувати його для використання на практиці при розв'язання задач математичного моделювання нелінійних процесів. Перспективи подальших досліджень можуть полягати у розробленні двобічних методів розв'язання задач для систем рівнянь з частинними похідними, рівнянь з частинними похідними вищих порядків та нестаціонарних багатовимірних задач, використовуючи напівдискретні методи (наприклад, метод прямих Роте).

КЛЮЧОВІ СЛОВА: задача Діріхле для напівлінійного еліптичного рівняння, додатний розв'язок, сильно інваріантний конусний відрізок, гетеротонний оператор, метод двобічних наближень, функція Гріна.

УДК 517.988 : 519.632

МЕТОД ДВУСТОРОННИХ ПРИБЛИЖЕНИЙ НА ОСНОВЕ ИСПОЛЬЗОВАНИЯ ФУНКЦИИ ГРИНА ПОСТРОЕНИЯ ПОЛОЖИТЕЛЬНОГО РЕШЕНИЯ ЗАДАЧИ ДИРИХЛЕ ДЛЯ ПОЛУЛИНЕЙНОГО ЭЛЛИПТИЧЕСКОГО УРАВНЕНИЯ

Гибкина Н. В. – канд. техн. наук, доцент, доцент каф. прикладной математики, Харьковский национальный университет радиоэлектроники, Харьков, Украина.

Ламтюгова С. Н. – канд. физ.-мат. наук, доцент, доцент каф. высшей математики, Харьковский национальный университет городского хозяйства имени А.Н. Бекетова, Харьков, Украина.

Сидоров М. В. – д-р физ.-мат. наук, доцент, профессор каф. прикладной математики, Харьковский национальный университет радиоэлектроники, Харьков, Украина.

АННОТАЦИЯ

Актуальность. Рассмотрен вопрос построения метода двусторонних приближений нахождения положительного решения задачи Дирихле для полулинейного эллиптического уравнения на основе использования метода функций Грина. Объектом исследования является первая краевая задача (задача Дирихле) для полулинейного эллиптического уравнения второго порядка.

Цель. Цель работы состоит в разработке на основе использования метода функций Грина метода двусторонних приближений решения задачи Дирихле для полулинейных эллиптических уравнений второго порядка и исследования его работы при решении тестовых задач.

Метод. С помощью метода функций Грина исходная первая краевая задача для полулинейного эллиптического уравнения заменяется эквивалентным интегральным уравнением Гаммерштейна. Интегральное уравнение представляется в виде нелинейного операторного уравнения с гетеротонным оператором и рассматривается в пространстве непрерывных функций, полуупорядоченном конусом неотрицательных функций. В качестве решения (обобщенного) краевой задачи принимается решение эквивалентного интегрального уравнения. Для гетеротонного оператора находится сильно инвариантный конусный отрезок, концы которого являются начальными приближениями для двух итерационных последовательностей. Первая из этих итерационных последовательностей является монотонно возрастающей и приближает искомое решение краевой задачи снизу, а вторая является монотонно убы-

DOI 10.15588/1607-3274-2021-3-3

вающей и приближает его сверху. Приведены условия существования единственного положительного решения рассматриваемой задачи Дирихле и двусторонней сходимости к нему последовательных приближений. Также приведены общие рекомендации по построению сильно инвариантного конусного отрезка. Разработанный метод имеет простую вычислительную реализацию и удобную для использования на практике апостериорную оценку погрешности.

Результаты. Разработанный метод программно реализован и исследован при решении тестовых задач. Результаты вычислительного эксперимента проиллюстрированы графической и табличной информациями.

Выводы. Проведенные эксперименты подтвердили работоспособность и эффективность разработанного метода и позволяют рекомендовать его для использования на практике при решении задач математического моделирования нелинейных процессов. Перспективы дальнейших исследований могут заключаться в разработке двусторонних методов решения задач для систем уравнений в частных производных, уравнений в частных производных высших порядков и нестационарных многомерных задач, используя полудискретные методы (например, метод прямых Роте).

КЛЮЧЕВЫЕ СЛОВА: задача Дирихле для полулинейного эллиптического уравнения, положительное решение, сильно инвариантный конусный отрезок, гетеротонный оператор, метод двусторонних приближений, функция Грина.

ЛІТЕРАТУРА / ЛИТЕРАТУРА

1. Pao C. V. Nonlinear parabolic and elliptic equations / C. V. Pao. – New York : Plenum Press, 1992. – 794 p. DOI: 10.1007/978-1-4615-3034-3
2. Samarskii A. A. Principles of Mathematical Modelling: Ideas, Methods, Examples / A. A. Samarskii, A. P. Samarskii. London : CRC Press, 2001 – 360 p. DOI: https://doi.org/10.1201/9781482288131
3. Kolmanovskii V. Models / V. Kolmanovskii, A. Myshkis // In: Applied Theory of Functional Differential Equations. Mathematics and Its Applications (Soviet Series). – 1992. – Vol. 85. – Dordrecht : Springer. – P. 1–34. DOI: 10.1007/978-94-015-8084-7_1
4. Frank-Kamenetskii D. A. Diffusion and Heat Exchange in Chemical Kinetics / D. A. Frank-Kamenetskii. – Princeton : Princeton University Press, 1955. – 384 p. DOI: 10.1515/9781400877195
5. Samarskii A. A. The Theory of Difference Schemes / A. A. Samarskii. – New York : CRC Press, 2001. – 786 p. DOI: 10.1201/9780203908518
6. Chen G. Algorithms and visualization for solutions of nonlinear elliptic equations / G. Chen, J. Zhou, W.-M. Ni // International Journal of Bifurcation and Chaos in Applied Sciences and Engineering. – 2000. – Vol. 10, No. 7. – P. 1565–1612. DOI: 10.1142/S0218127400001006
7. Kumar M. A finite element approach for finding positive solutions of semilinear elliptic Dirichlet problems / M. Kumar, P. Kumar // Numerical Methods for Partial Differential Equations: An International Journal. – 2009. – Vol. 25, № 5. – P. 1119–1128. DOI: 10.1002/num.20390
8. Kumar M. A review on nonlinear elliptic partial differential equations and approaches for solution / M. Kumar, G. Mishra // International Journal of Nonlinear Science. – 2012. – Vol. 13, № 4. – P. 401–418. DOI: IJNS.2012.06.30/621
9. Kumar M. An introduction to numerical methods for the solutions of partial differential equations / M. Kumar, G. Mishra // Applied Mathematics. – 2011. – Vol. 2, № 11. – P. 1327–1338. DOI: 10.4236/am.2011.211186
10. Matinfar M. A numerical extension on a convex nonlinear elliptic problem / M. Matinfar, K. Nemati // International Mathematical Forum. – 2008. – Vol. 3, № 17. – P. 811–816.
11. Yadav N. Artificial neural network technique for solution of nonlinear elliptic boundary value problems / N. Yadav, A. Yadav, K. Deep // Proceedings of Fourth International Conference on Soft Computing for Problem Solving. – New Delhi : Springer, 2015. – P. 113–121. DOI: 10.1007/978-81-322-2217-0_10
12. Kolosova S. V. On positive solutions of Liouville-Gelfand problem / S. V. Kolosova, V. S. Lukhanin, M. V. Sidorov // Вестник Казахского национального университета. Серия математика, механика, информатика. – 2018. – № 3 (99). – С. 78–91.
13. Konchakovska O. Numerical analysis of the one-dimensional nonlinear boundary value problem that modeling an electrostatic NEMS by two-sided approximations method / O. Konchakovska, M. Sidorov // Journal of Numerical Analysis, Industrial and Applied Mathematics (JNAIAM). – 2020. – Vol. 14, № 3–4. – P. 17–26.
14. Sidorov M. V. Green-Rvachev's quasi-function method for constructing two-sided approximations to positive solution of nonlinear boundary value problems // Carpathian Mathematical Publications. – 2018. – Vol. 10, № 2. – P. 360–375. DOI: 10.15330/cmp.10.2.360-375
15. Kantorovich L. V. Functional Analysis / L. V. Kantorovich, Akilov G. P. – New York : Pergamon, 1982. – 600 p. DOI: 10.1016/C2013-0-03044-7
16. Kantorovitch L. Linear operations in semi-ordered spaces. I / L. Kantorovitch // Matematicheskij sbornik. – 1940. – T. 7 (49), № 2. – P. 209–284.
17. Krasnosel'skii M. A. Positive Solutions of Operator Equations / M. A. Krasnosel'skii. – Groningen : Noordhoff, 1964. – 381 p.
18. Amann H. Fixed Point Equations and Nonlinear Eigenvalue Problems in Ordered Banach Spaces / H. Amann // SIAM Review. – 1976. – Vol. 18, № 4. – P. 620–709. DOI: 10.1137/1018114
19. Opojtsev V. I. A Generalization of the Theory of Monotone and Concave Operators / V. I. Opojtsev // Transactions of the Moscow Mathematical Society. – 1979. – Vol. 36. – P. 243–279.
20. Kurpel' N. S. On Some Modifications of Chaplygin's Method for Equations in Partially Ordered Spaces / N. S. Kurpel', V. I. Grechko // Ukrainian Mathematical Journal. – 1973. – Vol. 25. – P. 30–36. DOI: 10.1007/BF01085389
21. Shuvar B. A. Two-sided Operator Inequalities with Non-monotone Operators / B. A. Shuvar, M. I. Kopach // Differential Equations. – 2006. – Vol. 42. – P. 586–590. DOI: 10.1134/S001226610604015X
22. Kolosov A. I. A Boundary Value Problem on a Nonfixed Interval / A. I. Kolosov // Siberian Mathematical Journal. – 1976. – Vol. 17, № 6. – P. 944–948. DOI: https://doi.org/10.1007/BF00968020
23. Felipe-Sosa R. Boundary value problems for nonlinear elliptic equations with two parameters / R. Felipe-Sosa, R. Felipe // Differential Equations and Dynamical Systems. – 2014. – Vol. 22, № 1. – P. 1–13. DOI: 10.1007/s12591-013-0187-1
24. Franca M. Positive solutions of semilinear elliptic equations: a dynamical approach / M. Franca // Differential and Integral Equations. – 2013. – Vol. 26, № 5/6. – P. 505–554.
25. Rus M.-D. The method of monotone iterations for mixed monotone operators : Ph.D. Thesis Summary / M.-D. Rus. – Cluj-Napoca, 2010. – 45 p.
26. Sidorov M. V. Method of Two-Sided Approximations of the Solution of the First Boundary Value Problem for Nonlinear Ordinary Differential Equations Based on the Green's Function Use / M. V. Sidorov // Radio Electronics, Computer Science, Control. – 2019. – № 1 (48). – P. 57–66. DOI: 10.15588/1607-3274-2019-1-6
27. Vulikh B. Z. Introduction to the Theory of Partially Ordered Spaces / B. Z. Vulikh. – Groningen : Wolters-Noordhoff, 1967. – 403 p.

DOI 10.15588/1607-3274-2021-3-3