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RESTORATION OF DISCONTINUOUS FUNCTIONS BY DISCONTINUOUS INTERLINATION SPLINES

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ABSTRACT

Context. The problem of development and research of methods for approximation of discontinuous functions by discontinuous interlination splines and its further application to problems of computed tomography. The object of the study was the modeling of objects with a discontinuous internal structure.

Objective. The aim of this study is to develop a general method for constructing discontinuous interlining polynomial splines, which, as a special case, include discontinuous and continuously differentiated splines.

Method. Modern methods of restoring functions are characterized by new approaches to obtaining, processing and analyzing information. There is a need to build mathematical models in which information can be represented not only by function values at points, but also in the form of a set of function traces on planes or straight lines.

At the same time, practice shows that among the multidimensional objects that need to be investigated, more problems are described by a discontinuous functions.

The paper develops a general method for constructing discontinuous interlining polynomial splines, which, as a special case, include discontinuous and continuously differentiable splines. It is considered that the domain of the definition of the required two-dimensional function is divided into rectangular elements. Theorems on interlination and approximation properties of such discontinuous constructions are formulated and proved. The method is developed for approximating discontinuous functions of two variables based on the constructed discontinuous splines. The input data are the traces of an unknown function along a given system of mutually perpendicular straight lines. The proposed method has not only theoretical significance but also practical application in the IT domain, especially in computing tomography, allowing more accurately restore the internal structure of the body.

Results. The discontinuous interlination operator from known traces of the function of two variables on a system of mutually perpendicular straight lines is researched.

Conclusions. The functions of two variables that are discontinuous at some points or on some lines are better approximated by discontinuous spline interlinants. At the same time, equally high approximation estimates can be obtained. The results obtained have significant advantages over existing methods of interpolation and approximation of discontinuous functions. In further research, the authors plan to develop a theory of discontinuous splines on areas of complex shape bounded by arcs of known curves.

KEYWORDS: image processing, polynomial splines, interlination, discontinuous functions, approximation.

NOMENCLATURE

$\varphi 1_i^+(y)$ is a trace of function along the straight $x = x_i$ on the right;

$\varphi 1_i^-(y)$ is a trace of function along the straight $x = x_i$ on the left;

$f(x_k, y)$ is a trace of function on the line $x = x_k$;

$f(x, y_\ell)$ is a trace of function on the line $y = y_\ell$;

Π_{ij} is a rectangular element size $(x_{i-1}, x_i) \times (y_{j-1}, y_j)$;

$h1_{k,s}(x)$, $h2_{l,p}(x)$ are basic Hermitian polynomials of degree $2\rho - 1$ with properties;

$S_{ij}(x, y)$ is a discontinuous interlination polynomial spline, which corresponds to a given partition into rectangular elements Π_{ij};

$L_{ij}(x, y)$ is a discontinuous interpolation spline, which corresponds to a given partition into rectangular elements Π_{ij};

$C_{i,j}^{\pm\pm}$ is an one-sided function values at a point (x_i, y_j);

$C^{\mu,\nu}(D)$ is a class of two variables functions, which are defined and continuous in the domain and have continuous derivatives.

INTRODUCTION

The main attention in the theory of approximation of several variables functions by splines is given to the approximation of continuous and differentiable functions by continuous and differentiable splines ([1–3]), when using the least squares method [4, 5]. At the same time, the practice shows that among the multidimensional objects that need to be investigated, a larger number of problems are described by discontinuous functions. For example, in the methods of computed tomography, nowhere is information about the internal structure of the human body used (the stomach has one shape and the corresponding density of its tissues, the liver has a different shape and a different density of its tissues, the pancreas has its shape and density of tissues, the spine has its density, etc.)

In [6], it was proposed to use a priori information about parts for a more accurate description of the internal structure of a 3D body using the corresponding functions of three variables included in the equations where is the number of objects of the internal structure of the body to better restore it by computed tomography methods. In this method, it is proposed to use information about the inter-

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nal structure of the body in the form of a discontinuous function of three variables, which has discontinuities at the points of surfaces separating adjacent subregions.

The development of computational and applied mathematics suggests that the use of additional information about the object under study can lead to a more accurate restoration of this object. For example, in [7] it is proposed to use the equation of the surface of the human skull to more accurately restore the internal structure of the body.

In addition, we will give the following example. In solid mechanics, one of the most difficult problems is the problem of investigating cracks at the internal points of the body. It can be said that such a body has a discontinuous density: beyond the boundaries of the crack – one density, in the area bounded by the walls of the crack – another density.

The object of study is mathematical modeling of a discontinuous two-dimensional function by function interlination.

The function interlination operator allows you to restore a function of two variables with high accuracy. Traces functions on lines are used by the function interlining operator.

The subject of study is the theory of discontinuous intlinational splines.

The purpose of the work is to construct a discontinuous interlination spline to restore a two-dimensional function that has discontinuities of the first kind, when the sets of traces of the function on the lines are be used as function information.

1 PROBLEM STATEMENT

Let a discontinuous function of two variables $f(x,y)$ is given in the domain D. Suppose that the domain D is divided by straight lines $x_0=0<x_1<x_2<\ldots<x_m=1$, $y=y_0<y_1<y_2<\ldots<y_n=1$ into rectangular elements $\Pi_{ij}=(x_{i-1},x_i)\times(y_{j-1},y_j), i=\overline{1,m},\ j=\overline{1,n}$. The function $f(x,y)$ and its derivatives up to $\rho-1$ order have discontinuities of the first kind at the boundaries between these rectangular elements (not necessarily between all elements). It is required to construct a discontinuous spline such that the interlining and approximation properties are fulfilled.

2 REVIEW OF THE LITERATURE

Over the years, methods have been developed that approximate various important functions. These methods include Fourier series, Chebyshev series, Fourier-Jacobi and Pade-Jacobi polynomials, rational functions of Pade-Jacobi, Pade-Chebyshev and Pade-Legendre, as well as fractional and quasi-fractional approximations [8–11]. But these methods have a bad effect on the convergence of series when approximating functions with singularities. Loss of convergence occurs in the region with discontinuities and is called the Gibbs phenomenon. This phenomenon manifests itself in the vicinity of the discontinuity jumps and is an obstacle to the restoration of a discontinuous function. There are methods to reduce the Gibbs phenomenon [12, 13]. However, they do not completely remove it at all. A.L. Ageeva and T.V. Antonov proposed a method for determining the number of breakpoints and their positions based on the use of the Gibbs phenomenon [14, 15]. But the method requires more information: the smallest and largest values of the jumps of the approximate function. It is also assumed that the intervals in which the Gibbs phenomena arise do not intersect, i.e. it is impossible to separate the breakpoints that are close to each other.

That is, the development of the theory of approximation of discontinuous functions using discontinuous splines is a relevant task. This work belongs to a series of works by the authors aimed at the study and improvement of mathematical models in computed tomography [16–18]. To date, tomography has developed many computational methods, algorithms and software tools aimed at restoring the internal properties of an object. They perform well when restoring objects with smooth properties, but give unsatisfactory results for objects with discontinuous characteristics. Therefore, there is a need to create mathematical methods for approximating discontinuous functions for a more accurate idea of the structure of the studied object.

A series of works by authors [19, 20] devoted to solving the flat problem of Radon computed tomography using the heterogeneity of the internal structure of a two-dimensional body. For this purpose, it is advisable to use function interlination operators, since these operators restore (possibly approximated) functions on their known traces on a given system of lines. They provide an opportunity to construct operators whose integrals from these lines (linear integrals) will be equal to integrals from the most renewable function. That is, interlination is a mathematical apparatus, naturally related to the task of restoring the characteristics of objects according to their known projections. This article is a continuation of this article series.

In this article, we construct the discontinuous interlineation operator from known traces of the function of two variables on a system of mutually perpendicular straight lines.

3 MATERIALS AND METHODS

Let us introduce the notation: $\varphi 1_i^+(y)=\lim\limits_{x\to x_i+0} f(x,y)$, $\varphi 1_i^-(y)=\lim\limits_{x\to x_i-0} f(x,y)$ – function traces on the straight $x=x_i,\ i=\overline{1,m}$. If $\varphi 1_i^+(y)=\varphi 1_i^-(y)$, then the function $f(x,y)$ is continuous on the line $x=x_i$, otherwise it has a break on the given line. Consider the element $\Pi_{ij}=(x_{i-1},x_i)\times(y_{j-1},y_j)$, $i=\overline{1,m},\ j=\overline{1,n}$.

Definition. We will call a discontinuous interlination polynomial spline in a domain D, which corresponds to a

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given partition into subdomains Π_{ij}, the following function

$$S(x,y)=S_{ij}(x,y),\ (x,y)\in\Pi_{ij}$$

$$S_{ij}(x,y)=S1_{ij}(x,y)+S2_{ij}(x,y)-S12_{ij}(x,y),\quad (x,y)\in\Pi_{ij}\subset D \tag{1}$$

where

$$S1_{ij}(x,y)= = S1_{ij}(x,y;\{\varphi1_{i-1,s}(y)\};\{\varphi1_{i,s}(y)\},s=\overline{0,\rho-1})= =\sum_{s=0}^{\rho-1}\varphi1^{+}_{i-1,s}(y)\cdot h1_{i-1,s}(x)+\sum_{s=0}^{\rho-1}\varphi1^{-}_{i,s}(y)\cdot h1_{i,s}(x);$$

$$S2_{ij}(x,y)= = S2_{ij}(x,y;\{\varphi2_{j-1,p}(x)\};\{\varphi2_{j,p}(x)\},p=\overline{0,\rho-1})= =\sum_{p=0}^{\rho-1}\varphi2^{+}_{j-1,p}(x)\cdot h2_{j-1,p}(y)+\sum_{p=0}^{\rho-1}\varphi2^{+}_{j,p}(x)\cdot h2_{j,p}(y);$$

$$S12_{ij}(x,y)= = S12_{ij}(x,y;\{\varphi1_{i-1,s}(y)\};\{\varphi1_{i,s}(y)\},s=\overline{0,\rho-1}, \{\varphi2_{j-1,p}(x)\};\{\varphi2_{j,p}(x)\},p=\overline{0,\rho-1})=$$
$$=\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{++}_{i-1,j-1,s,p}h1_{i-1,s}(x)h2_{j-1,p}(y)+\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{+-}_{i-1,j,s,p}h1_{i-1,s}(x)h2_{j,p}(y)+\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{-+}_{i,j-1,s,p}h1_{i,s}(x)h2_{j-1,p}(y)+\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{--}_{i,j,s,p}h1_{i,s}(x)h2_{j,p}(y),$$

$h1_{k,s}(x)$, $h2_{l,p}(x)$ – basic Hermitian polynomials of degree $2\rho-1$ with properties:

$$h1^{(s')}{}_{k,s}(x_{k'})=\delta_{k,k'}\delta_{s,s'},\ k,k'\in\{i-1,i\},s,s'\in\{0,\rho-1\},$$
$$h2^{(p')}{}_{l,p}(y_{l'})=\delta_{l,l'}\delta_{p,p'},l,l'\in\{j-1,j\},p,p'\in\{0,\rho-1\}.$$

Theorem 1. If

$$\varphi1^{+(p)}_{i,s}(y_j)=\varphi2^{+(s)}_{j,p}(x_i)=C^{++}{}_{ijsp},$$
$$\varphi1^{-(p)}_{i,s}(y_j)=\varphi2^{+(s)}_{j,p}(x_i)=C^{-+}{}_{ijsp},$$
$$\varphi1^{-(p)}_{i,s}(y_j)=\varphi2^{-(s)}_{j,p}(x_i)=C^{--}{}_{ijsp},$$
$$\varphi1^{+(p)}_{i,s}(y_j)=\varphi2^{-(s)}_{j,p}(x_i)=C^{+-}{}_{ijsp}$$

then on the border of the rectangle Π_{ij} the function $S_{ij}(x,y)$ satisfies the following relations

$$\left.\frac{\partial^{s'}S_{ij}(x,y)}{\partial x^{s'}}\right|_{x=x_{i-1}}=\varphi1^{+}_{i-1,s'}(y),\quad \left.\frac{\partial^{s'}S_{ij}(x,y)}{\partial x^{s'}}\right|_{x=x_{i}}=\varphi1^{-}_{i,s'}(y),\quad y_{j-1}\le y\le y_j,\ s'=\overline{0,\rho-1} \tag{2}$$

$$\left.\frac{\partial^{p'}S_{ij}(x,y)}{\partial y^{p'}}\right|_{y=y_{j-1}}=\varphi2^{+}_{j-1,p'}(x),\quad \left.\frac{\partial^{p'}S_{ij}(x,y)}{\partial y^{p'}}\right|_{y=y_{j}}=\varphi2^{-}_{j,p'}(x),\quad x_{i-1}\le x\le x_i,\ p'=\overline{0,\rho-1} \tag{3}$$

Proof. Substitute in formula (1) $x=x_{i-1}$. As a result, we get $S_{ij}(x_{i-1},y)=$

$$=S1_{ij}(x_{i-1},y)+S2_{ij}(x_{i-1},y)-S12_{ij}(x_{i-1},y)=$$
$$=\sum_{s=0}^{\rho-1}\varphi1^{+}_{i-1,s}(y)\cdot h1_{i-1,s}(x_{i-1})+\sum_{s=0}^{\rho-1}\varphi1^{-}_{i,s}(y)\cdot h1_{i,s}(x_{i-1})+\sum_{p=0}^{\rho-1}\varphi2^{+}_{j-1,p}(x_{i-1})\cdot h2_{j-1,p}(y)+\sum_{p=0}^{\rho-1}\varphi2^{-}_{j,p}(x_{i-1})\cdot h2_{j,p}(y)-$$
$$-\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{++}_{i-1,j-1,s,p}h1_{i-1,s}(x_{i-1})h2_{j-1,p}(y)+\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{+-}_{i-1,j,s,p}h1_{i-1,s}(x_{i-1})h2_{j,p}(y)+\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{-+}_{i,j-1,s,p}h1_{i,s}(x_{i-1})h2_{j-1,p}(y)+\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{--}_{i,j,s,p}h1_{i,s}(x_{i-1})h2_{j,p}(y)=$$
$$=\sum_{s=0}^{\rho-1}\varphi1^{+}_{i-1,s}(y)\cdot\delta_{i-1,i-1}\delta_{s,0}+\sum_{s=0}^{\rho-1}\varphi1^{-}_{i,s}(y)\cdot\delta_{i,i-1}\delta_{s,0}+\sum_{p=0}^{\rho-1}\varphi2^{+}_{j-1,p}(x_{i-1})\cdot h2_{j-1,p}(y)+$$

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$$+\sum_{p=0}^{\rho-1}\varphi 2^{-}_{j,p}(x_{i-1})\cdot h2_{j,p}(y)-$$

$$-\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{++}_{i-1,j-1,s,p}\delta_{i-1,i-1}\delta_{s,0}h2_{j-1,p}(y)+$$

$$+\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{+-}_{i-1,j,s,p}\delta_{i-1,i-1}\delta_{s,0}h2_{j,p}(y)+$$

$$+\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{-+}_{i,j-1,s,p}\delta_{i,i-1}\delta_{s,0}h2_{j-1,p}(y)+$$

$$+\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{--}_{i,j,s,p}\delta_{i,i-1}\delta_{s,0}h2_{j,p}(y)=$$

$$=\varphi 1^{+}_{i-1,0}(y)+\sum_{p=0}^{\rho-1}\varphi 2^{+}_{j-1,p}(x_{i-1})\cdot h2_{j-1,p}(y)+$$

$$+\sum_{p=0}^{\rho-1}\varphi 2^{-}_{j,p}(x_{i-1})\cdot h2_{j,p}(y)-$$

$$-\sum_{p=0}^{\rho-1}C^{++}_{i-1,j-1,0,p}h2_{j-1,p}(y)-\sum_{p=0}^{\rho-1}C^{+-}_{i-1,j,0,p}h2_{j,p}(y)=$$

$$=\left|\begin{matrix}C^{++}_{i-1,j-1,0,p}=\varphi 2^{+}_{j-1,p}(x_{i-1})\\ C^{+-}_{i-1,j,0,p}=\varphi 2^{-}_{j,p}(x_{i-1})\end{matrix}\right|=$$

$$=\varphi 1^{+}_{i-1,0}(y)+\sum_{p=0}^{\rho-1}\varphi 2^{+}_{j-1,p}(x_{i-1})\cdot h2_{j-1,p}(y)+$$

$$+\sum_{p=0}^{\rho-1}\varphi 2^{-}_{j,p}(x_{i-1})\cdot h2_{j,p}(y)-$$

$$-\sum_{p=0}^{\rho-1}\varphi 2^{+}_{j-1,p}(x_{i-1})h2_{j-1,p}(y)-$$

$$-\sum_{p=0}^{\rho-1}\varphi 2^{-}_{j,p}(x_{i-1})h2_{j,p}(y)=\varphi 1^{+}_{i-1,0}(y)$$

Thus, we have proved that $S_{ij}(x_{i-1},y)=\varphi 1^{+}_{i-1,0}(y)$, $y_{j-1}\le y\le y_j$.

The equalities are proved similarly when we substitute into formula (1) $x=x_i$, $y=y_{j-1}$, $y=y_j$.

Let us assume that $1\le s'\le \rho-1$. As a result, we get

$$\left.\frac{\partial^{s'}S_{ij}(x,y)}{\partial x^{s'}}\right|_{x=x_i}=\left.\frac{\partial^{s'}S1_{ij}(x,y)}{\partial x^{s'}}\right|_{x=x_i}+$$

$$+\left.\frac{\partial^{s'}S2_{ij}(x,y)}{\partial x^{s'}}\right|_{x=x_i}-\left.\frac{\partial^{s'}S12_{ij}(x,y)}{\partial x^{s'}}\right|_{x=x_i}=$$

$$=\sum_{s=0}^{\rho-1}\varphi 1^{+}_{i-1,s}(y)\cdot\left.\frac{\partial^{s'}}{\partial x^{s'}}h1_{i-1,s}(x)\right|_{x=x_{i-1}}+$$

$$+\sum_{s=0}^{\rho-1}\varphi 1^{-}_{i,s}(y)\cdot\left.\frac{\partial^{s'}}{\partial x^{s'}}h1_{i,s}(x)\right|_{x=x_{i-1}}+$$

$$+\sum_{p=0}^{\rho-1}\left.\frac{\partial^{s'}}{\partial x^{s'}}\varphi 2^{+}_{j-1,p}(x)\right|_{x=x_{i-1}}\cdot h2_{j-1,p}(y)+$$

$$+\sum_{p=0}^{\rho-1}\left.\frac{\partial^{s'}}{\partial x^{s'}}\varphi 2^{-}_{j,p}(x)\right|_{x=x_{i-1}}\cdot h2_{j,p}(y)-$$

$$-\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{++}_{i-1,j-1,s,p}\left.\frac{\partial^{s'}}{\partial x^{s'}}h1_{i-1,s}(x)\right|_{x=x_{i-1}}\cdot h2_{j-1,p}(y)-$$

$$-\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{+-}_{i-1,j,s,p}\left.\frac{\partial^{s'}}{\partial x^{s'}}h1_{i-1,s}(x)\right|_{x=x_{i-1}}\cdot h2_{j,p}(y)-$$

$$-\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{-+}_{i,j-1,s,p}\left.\frac{\partial^{s'}}{\partial x^{s'}}h1_{i,s}(x)\right|_{x=x_{i-1}}\cdot h2_{j-1,p}(y)+$$

$$+\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{--}_{i,j,s,p}\left.\frac{\partial^{s'}}{\partial x^{s'}}h1_{i,s}(x)\right|_{x=x_{i-1}}\cdot h2_{j,p}(y)=$$

$$=\sum_{s=0}^{\rho-1}\varphi 1^{+}_{i-1,s}(y)\cdot\delta_{i-1,i-1}\delta_{s',s}+\sum_{s=0}^{\rho-1}\varphi 1^{-}_{i,s}(y)\cdot\delta_{i,i-1}\delta_{s',s}+$$

$$+\sum_{p=0}^{\rho-1}\varphi 2^{+(s')}_{j-1,p}(x_{i-1})\cdot h2_{j-1,p}(y)+$$

$$+\sum_{p=0}^{\rho-1}\varphi 2^{-(s')}_{j,p}(x_{i-1})\cdot h2_{j,p}(y)-$$

$$-\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{++}_{i-1,j-1,s,p}\delta_{i-1,i-1}\delta_{s',s}h2_{j-1,p}(y)-$$

$$-\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{+-}_{i-1,j,s,p}\delta_{i-1,i-1}\delta_{s',s}h2_{j,p}(y)-$$

$$-\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{-+}_{i,j-1,s,p}\delta_{i,i-1}\delta_{s',s}h2_{j-1,p}(y)+$$

$$+\sum_{s=0}^{\rho-1}\sum_{p=0}^{\rho-1}C^{--}_{i,j,s,p}\delta_{i,i-1}\delta_{s',s}h2_{j,p}(y)=$$

$$=\left|C^{++}_{i-1,j-1,s',p}=\varphi 2^{+(s')}_{j-1,p}(x_{i-1})\right|=\varphi 1^{+}_{i-1,s'}(y)+$$

$$+\sum_{p=0}^{\rho-1}C^{++}_{i-1,j-1,s',p}\cdot h2_{j-1,p}(y)+\sum_{p=0}^{\rho-1}C^{-+}_{i-1,j,s',p}\cdot h2_{j,p}(y)-$$

$$-\sum_{p=0}^{\rho-1}C^{++}_{i-1,j-1,s',p}h2_{j-1,p}(y)-\sum_{p=0}^{\rho-1}C^{+-}_{i-1,j,s',p}h2_{j,p}(y)=$$

$$=\varphi 1^{+}_{i-1,s'}(y).$$

Properties (2) with $x=x_i$ and properties (3) are proved similarly.

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Theorem 1 is proved.

Theorem 1. If $\varphi 1^{-}_{i,s}(y) = \varphi 1^{+}_{i,s}(y) = \varphi 1_{i,s}(y)$, $s = \overline{0,\mu}, 0 \le \mu \le \rho - 1$, $\varphi 2^{-}_{j,p}(x) = \varphi 2^{+}_{j,p}(x) = \varphi 2_{j,p}(x)$, $p = \overline{0,\nu}, 0 \le \nu \le \rho - 1$, then the function $S(x,y) = S_{ij}(x,y)$, $(x,y) \in \Pi_{ij}$ will have properties like this

$$S(x,y) \in C^{\mu,\nu}(D),$$

$$\left. \frac{\partial^{s'} S(x,y)}{\partial x^{s'}} \right|_{x=x_i} = \varphi 1_{i,s'}(y), \tag{4}$$

$$i = \overline{1,m},\ s' = \overline{0,\mu},\ y_{j-1} \le y \le y_j,$$

$$\left. \frac{\partial^{p'} S(x,y)}{\partial y^{p'}} \right|_{y=y_j} = \varphi 2_{j,p'}(x), \tag{5}$$

$$j = \overline{1,n},\ p' = \overline{0,\nu},\ x_{i-1} \le x \le x_i.$$

The proof follows from the fact that if functions $\varphi 1_{i,s}(y) \in C^{\rho-1}[x_{i-1}, x_i]$, $\varphi 2_{j,p}(x) \in C^{\rho-1}[y_{j-1}, y_j]$, then in each element Π_{ij} the function $S_{ij}(x,y)$ will belong to the class $C^{\rho-1,\rho-1}(\Pi_{ij})$.Thus, the function $S(x,y)$ in each of the elements Π_{ij} belongs to the class $C^{\rho-1,\rho-1}(\Pi_{ij})$ and on the boundary between the neighboring Π_{ij} elements, it preserves the continuity of the derivatives up to orders μ, ν, respectively, since the proof of properties (4), (5) is carried out by analogy with the proof of properties in Theorem 1.

Theorem 2 is proved.

Remark 1. If the conditions of Theorem 2 are satisfied, then the function $S(x,y)$ has discontinuous partial derivatives of orders greater than μ in x and larger than ν in y, respectively.

Remark 2. In principle, it is assumed that discontinuities of a function $S(x,y)$ and its partial derivatives up to the corresponding orders can exist only at the boundaries of one or several elements.

Theorem 3. If the functions $\varphi 1^{+}_{i,s}(y), \varphi 1^{-}_{i,s}(y)$ are polynomials (generally speaking, different) of degree $Q \ge 2\rho - 1$ and the functions $\varphi 2^{+}_{j,p}(x), \varphi 2^{-}_{j,p}(x)$ are polynomials (generally speaking, different) of degree $Q \ge 2\rho - 1$, then the function $S(x,y)$ will be a piecewise polynomial discontinuous spline, which is a polynomial in two variables on each rectangle $\Pi_{ij} \subset D$. In particular, if $Q = 2\rho - 1$, then $S(x,y)$ will be a discontinuous piecewise polynomial spline of (x,y) degree $2\rho - 1$ in each variable.

The proof follows from the fact that the functions $S_{ij}(x,y)$ use the Hermitian polynomial basis functions and will be polynomials in the assumptions of Theorem 3. If $Q = 2\rho - 1$, then $S_{ij}(x,y)$ will be a polynomial of degree $2\rho - 1$ in each variable. If the conditions of Theorem 2 are not satisfied, then such a function $S_{ij}(x,y)$ will have gaps between different elements.

Remark 3. We emphasize again that these gaps may not necessarily be at the border between all elements. Moreover, it is not required that on all four sides of each element the spline has discontinuous derivatives of orders $\mu+1,\ \mu+2,...,\rho-1$ and $\nu+1, \nu+2,...,\rho-1$ concerning x and y, respectively.

Theorem 4. Let us assume that the function to be approximated

$$f(x,y) \in C^{\rho-1,\rho-1}(D \setminus \overline{\Pi_{kl}}),$$

$$\varphi 1^{+}_{i-1,s}(y) \ne \varphi 1^{-}_{i,s}(y),$$

$$\varphi 2^{+}_{j-1,p}(x) \ne \varphi 2^{-}_{j,p}(x),\ s,p = \overline{0,\rho-1}.$$

Then, if in $S(x,y)$ substitute

$$\varphi 1^{-}_{i',s}(y) = \varphi 1^{+}_{i',s}(y) = f^{(s',0)}(x_{i'}, y),$$

$$i' \in \{0,1,...,m\},\ i' \ne i-1,\ i' \ne i,\ 0 \le y \le 1;$$

$$\varphi 2^{-}_{j',p}(x) = \varphi 2^{+}_{j',p}(x) = f^{(0,p)}(x, y_{j'}),$$

$$j' \in \{0,1,...,n\},\ j' \ne j-1,\ j' \ne j,\ 0 \le x \le 1;$$

$$\varphi 1^{-}_{i-1,s}(y) = \varphi 1^{+}_{i-1,s}(y) = f^{(s,0)}(x_{i-1}, y),$$

$$0 \le y \le y_{j-1} \text{ or } y_j \le y \le 1;$$

$$\varphi 2^{-}_{j-1,p}(x) = \varphi 2^{+}_{j-1,p}(x) = f^{(0,p)}(x, y_{j-1}),$$

$$0 \le x \le x_{i-1} \text{ or } x_i \le x \le 1,$$

$$\varphi 1^{+}_{i-1,s}(y) = f^{(s,0)}(x_{i-1}+0, y),\ \varphi 1^{-}_{i,s}(y) = f^{(s,0)}(x_i - 0, y),$$

$$\varphi 1^{-}_{i-1,s}(y) = f^{(s,0)}(x_{i-1}-0, y),\ \varphi 1^{+}_{i,s}(y) = f^{(s,0)}(x_i + 0, y),$$

$$\varphi 2^{+}_{j-1,p}(x) = f^{(0,p)}(x, y_{j-1}+0),$$

$$\varphi 2^{-}_{j,p}(x) = f^{(0,p)}(x, y_j - 0),$$

$$\varphi 2^{-}_{j-1,p}(x) = f^{(0,p)}(x, y_{j-1}-0),$$

$$\varphi 2^{+}_{j,p}(x) = f^{(0,p)}(x, y_j + 0),$$

then the resulting function $S(x,y)$ will belong to the class $C^{\rho-1,\rho-1}(D)$ and will be discontinuous along with its derivatives up to the order $\rho - 1$ in each variable only on the element boundary Π_{ij}.

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The proof follows from the fact that on the boundary between all elements (except for the element Π_{ij}) the function $S(x,y)$ will have continuous derivatives up to the order $\rho-1$ inclusive and only on the boundary of the element Π_{ij} can it be discontinuous together with its partial derivatives. That is, such a function will belong to the class $S(x,y)\in C^{\rho-1,\rho-1}(D\setminus\overline{\Pi_{kl}})$.

Theorem 5. If the conditions of Theorem 4 are satisfied, then for the error of approximation of such a discontinuous function $f(x,y)$, the corresponding discontinuous interlinational polynomial spline $S(x,y)$ will satisfy the following relation:

$$\left|f(x,y)-S(x,y)\right|=O(\Delta 1^{2\rho}\Delta 2^{2\rho}),\ (x,y)\in\Pi_{kl}\neq\Pi_{i,j},$$
$$\Delta 1=\max_k(x_k-x_{k-1}),\ \Delta 2=\max_l(y_l-y_{l-1}),$$
$$\left|f(x,y)-S(x,y)\right|=O(\Delta i^{2\rho}\Delta j^{2\rho}),\ (x,y)\in\Pi_{i,j},$$
$$\Delta i=x_i-x_{i-1},\ \Delta j=y_j-y_{j-1},\ (i,j)\neq(k,l)$$

provided that $f(x,y)\in C^{\rho-1,\rho-1}(\Pi_{i,j})$.

Proof. The operator $S_{ij}(x,y)=S_{ij}f(x,y)$ according to definition 1 can be written as $S_{ij}f(x,y)=S1_{ij}f(x,y)+$ $+S2_{ij}f(x,y)-S12_{ij}f(x,y)$.

According to Theorem 3.2.1 in [3], the remainder of the approximation by the interlination formulas is expressed as the operator product of the remainders of the approximation of a function $f(x,y)$ by the operators $S1_{ij}f(x,y)$ and $S2_{ij}f(x,y)$

$$\begin{aligned}RS_{ij}f(x,y)&=(f(x,y)-S_{ij}f(x,y))=\\&=(f(x,y)-S1_{ij}f(x,y)-S2_{ij}f(x,y)+S12_{ij}f(x,y))=\\&=(f(x,y)-S1_{ij}f(x,y))(f(x,y)-S2_{ij}f(x,y))=\\&=RS1_{ij}f(x,y)RS2_{ij}f(x,y).\end{aligned}$$

Theorem 5 is proved.

4 EXPERIMENTS

Let be $\rho=1,\ m=2,\ n=2$. Let's set the nodes: $x_0=0,\ x_1=0.5, x_2=1,\ y_0=0,\ y_1=0.5, y_2=1$.

That is, the domain of definition of the function being approximated (Figure 1) consists of four rectangular elements, which are set as follows:

$$\Pi_{11}=\{(x,y): x_0<x<x_1, y_0<y<y_1\},$$
$$\Pi_{12}=\{(x,y): x_0<x<x_1, y_1<y<y_2\},$$
$$\Pi_{21}=\{(x,y): x_1<x<x_2, y_0<y<y_1\},$$
$$\Pi_{11}=\{(x,y): x_1<x<x_2, y_1<y<y_2\}.$$

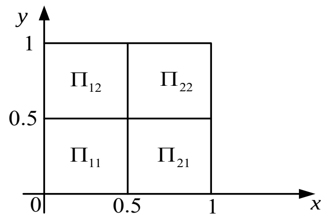

Figure 1 – Domain of the function to be approximated $f(x,y)$

Let's define a function at the corner points of the elements:

$$\Pi_{11}:\left|\begin{array}{l}f^{+,+}(0;0)=f(0+0;0+0)=1,\\ f^{+,-}(0;0.5)=f(0+0;0.5-0)=2,\\ f^{-,-}(0.5;0.5)=f(0.5-0;0.5-0)=1,\\ f^{-,+}(0.5;0)=f(0.5-0;0+0)=2,\end{array}\right.$$

$$\Pi_{12}:\left|\begin{array}{l}f^{+,+}(0;0.5)=f(0+0;0.5+0)=1,\\ f^{+,-}(0;1)=f(0+0;1-0)=2,\\ f^{-,-}(0.5;1)=f(0.5-0;1-0)=1,\\ f^{-,+}(0.5;0.5)=f(0.5-0;0.5+0)=2,\end{array}\right.$$

$$\Pi_{22}:\left|\begin{array}{l}f^{+,+}(0.5;0.5)=f(0.5+0;0.5+0)=3,\\ f^{+,-}(0.5;1)=f(0.5+0;1-0)=4,\\ f^{-,-}(1;1)=f(1-0;1-0)=3,\\ f^{-,+}(1;0.5)=f(1-0;0.5+0)=4,\end{array}\right.$$

$$\Pi_{21}:\left|\begin{array}{l}f^{+,+}(0.5;0)=f(0.5+0;0+0)=3,\\ f^{+,-}(0.5;0.5)=f(0.5+0;0.5-0)=4,\\ f^{-,-}(1;0.5)=f(1-0;0.5-0)=3,\\ f^{-,+}(1;0)=f(1-0;0+0)=4.\end{array}\right.$$

The discontinuous spline will be constructed in the form (1)

$$S(x,y)=\begin{cases}2x+2y-8xy+1, & (x,y)\in\Pi_{11},\\ -10x-6y+8+8xy, & (x,y)\in\Pi_{12},\\ -4xy+2x+2y+2\ , & (x,y)\in\Pi_{21},\\ -8xy+6x+6y-1, & (x,y)\in\Pi_{22}.\end{cases}$$

As you can see, the function $S(x,y)$ at the border between the elements Π_{11} and Π_{21} at $x<x_1$ will have the following traces:

$$S(x_1-0,y)=S_{11}(x_1,y)=f^{-,+}(0.5;0)\frac{y-y_1}{y_0-y_1}+$$
$$+f^{-,-}(0.5;0.5)\frac{y-y_0}{y_1-y_0},\ y_0\le y\le y_1.$$

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Similarly, $S(x_1+0,y)=S_{21}(x_1,y)=f^{+,+}(0.5;0)\frac{y-y_1}{y_0-y_1}+$

$$+f^{+,-}(0.5;0.5)\frac{y-y_0}{y_1-y_0},\ y_0\le y\le y_1.$$

That is, if $f^{-,+}(0.5,0)\ne f^{+,+}(0.5,0)$, then at a point $(0.5;0)$ such a spline will be discontinuous. In addition, if at a point $f^{+,+}(0.5;0.5)\ne f^{+,-}(0.5;0.5)$, then the spline will be discontinuous along the entire line $x=0.5,\ \ y_0\le y\le y_1$.

Let us define the function to be approximated in the form

$$f(x,y)=S_{ij}(x,y)+$$

$$+\frac{(x-x_{i-1})(x_i-x)(y-y_{j-1})(y_j-y)}{4},(x,y)\in\Pi_{i,j},\ i,j=1,2.$$

Thus, in each of the four elements of the assignment, the function to be approximated has a partial derivative $f^{2,2}(x,y)\equiv 1$, $\forall(x,y)\in\Pi_{ij}$. Therefore, according to the theory, the error in the approximation of such a discontinuous function written above by a discontinuous spline will satisfy the inequality:

$$\max_{(x,y)\in\Pi_{ij}}\left|f(x,y)-S_{i,j}(x,y)\right|\le f^{(2,2)}(\xi,\eta)\cdot\frac{\Delta i^2\Delta j^2}{2!\cdot 2!}=$$

$$=1\cdot\frac{\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2}{2!\cdot 2!}=\frac{1}{64}\approx 0.016.$$

Suppose that in the domain defined in the previous example, a function with discontinuities of the first kind in the nodes of a given rectangular grid is given (Fig. 2):

$$f(x,y)=\begin{cases}x^2+y^2, & (x,y)\in\Pi_{11};\\ x^2-y^2, & (x,y)\in\Pi_{12};\\ -x^2+y^2, & (x,y)\in\Pi_{21};\\ -x^2-y^2, & (x,y)\in\Pi_{22}.\end{cases}$$

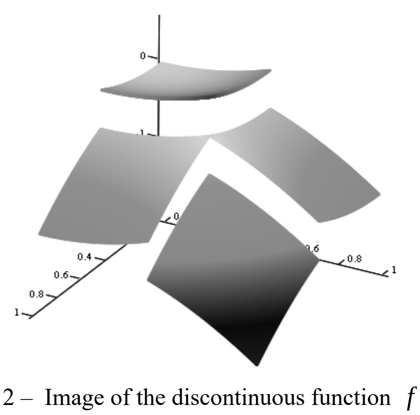

Figure 2 – Image of the discontinuous function $f(x,y)$

This function has discontinuities of the first kind at the boundaries of a given rectangular grid, and hence at the corner points of the grid.

First, we construct a discontinuous bilinear approximation spline on a given rectangular grid, for which we use the formula of the approximation spline [18] in each element of the partition

$$L_{ij}(x,y)=C_{i-1,j-1}^{++}\frac{x-x_i}{x_{i-1}-x_i}\frac{y-y_j}{y_{j-1}-y_j}+$$

$$C_{i-1,j}^{+-}\frac{x-x_i}{x_{i-1}-x_i}\frac{y-y_{j-1}}{y_j-y_{j-1}}+C_{i,j-1}^{-+}\frac{x-x_{i-1}}{x_i-x_{i-1}}\frac{y-y_j}{y_{j-1}-y_j}+$$

$$+C_{i,j}^{--}\frac{x-x_{i-1}}{x_i-x_{i-1}}\frac{y-y_{j-1}}{y_j-y_{j-1}},\ \ (x,y)\in\Pi_{ij},$$

where C – matrix of unknown coefficients.

In this case, the experimental data are the values of the function at the corner points of the rectangular grid, i.e.

$$f^{-+}(0.5;0)=0.25,\ \ f^{++}(0.5;0)=-0.25,$$
$$f^{+-}(0;0.5)=0.25,\ \ f^{++}(0;0.5)=-0.25,$$
$$f^{--}(0.5;0.5)=0.5,\ \ f^{-+}(0.5;0.5)=0,$$
$$f^{++}(0.5;0.5)=-0.5,\ \ f^{+-}(0.5;0.5)=0,$$
$$f^{--}(1;0.5)=-0.75,\ \ f^{-+}(1;0.5)=-1.25,$$
$$f^{--}(0.5;1)=-0.75,\ \ f^{+-}(0.5;1)=-1.25.$$

Next, using the method of least squares, we solve the minimization problem:

$$F(C)=\iint_D(f(x,y)-L(x,y,C))^2\,dxdy\to\min.$$

This problem was solved in the computer mathematics system MathCad and the following matrix of coefficients was obtained:

$$C=\begin{pmatrix}-0.083 & 0.167 & 0.167 & 0.417\\ -0.25 & 0 & -1 & -0.75\\ -0.25 & -1 & 0 & -0.75\\ -0.417 & -1.167 & -1.167 & -1.917\end{pmatrix}.$$

That is, the bilinear approximating spline takes the form (Fig. 3).

$$L(x,y)=\begin{cases}0.5x+0.5y-0.083, & (x,y)\in\Pi_{11};\\ 0.5x-1.5y+0.5, & (x,y)\in\Pi_{12};\\ -1.5x+0.5y+0.5, & (x,y)\in\Pi_{21};\\ -1.5x-1.5y+1.083, & (x,y)\in\Pi_{22}.\end{cases}$$

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Now on a given grid of nodes for a given discontinuous function $f(x,y)$ we construct a discontinuous interlination spline in the form of formula (1), which for our case takes the form

$$S(x,y)=S_{ij}(x,y),\ (x,y)\in\Pi_{ij},$$

$$S_{ij}(x,y)=S1_{ij}(x,y)+S2_{ij}(x,y)-$$
$$-S12_{ij}(x,y),(x,y)\in\Pi_{ij}\subset D,$$

$$S1_{ij}(x,y)=\varphi1^{+}{}_{i-1}(y)\cdot\frac{x-x_i}{x_{i-1}-x_i}+\varphi1^{-}{}_{i}(y)\cdot\frac{x-x_{i-1}}{x_i-x_{i-1}};$$

$$S2_{ij}(x,y)=\varphi2^{+}{}_{j-1}(x)\cdot\frac{y-y_j}{y_{j-1}-y_j}+\varphi2^{-}{}_{j}(x)\cdot\frac{y-y_{j-1}}{y_j-y_{j-1}};$$

$$S12_{ij}(x,y)=\varphi1^{+}{}_{i-1}(y)\cdot\frac{x-x_i}{x_{i-1}-x_i}+\varphi1^{-}{}_{i}(y)\cdot\frac{x-x_{i-1}}{x_i-x_{i-1}};$$

$$S12_{ij}(x,y)=C^{++}_{i-1,j-1}\frac{x-x_i}{x_{i-1}-x_i}\frac{y-y_j}{y_{j-1}-y_j}+$$
$$+C^{+-}_{i-1,j}\frac{x-x_i}{x_{i-1}-x_i}\frac{y-y_{j-1}}{y_j-y_{j-1}}+C^{-+}_{i,j-1}\frac{x-x_{i-1}}{x_i-x_{i-1}}\frac{y-y_j}{y_{j-1}-y_j}+$$
$$+C^{--}_{i,j}\frac{x-x_{i-1}}{x_i-x_{i-1}}\frac{y-y_{j-1}}{y_j-y_{j-1}};$$

where

$$C^{++}_{i-1\,j-1}=\lim_{\substack{x\to x_{i-1}+0\\ y\to y_{j-1}+0}}f(x,y),\qquad C^{+-}_{i-1\,j}=\lim_{\substack{x\to x_{i-1}+0\\ y\to y_j-0}}f(x,y),$$
$$C^{-+}_{ij-1}=\lim_{\substack{x\to x_i-0\\ y\to y_{j-1}+0}}f(x,y),\qquad C^{--}_{ij}=\lim_{\substack{x\to x_i-0\\ y\to y_j-0}}f(x,y).$$

To do this, the experimental data will be traces of the passed function along a given system of lines $x=x_0=0,\ x=x_1=0.5,\ x=x_2=1,$
$y=y_0=0,\ y=y_1=0.5,\ y=y_2=1$, namely,

$\Pi_{11}:\ \varphi1_0^{+}(y)=\lim\limits_{\substack{x\to+0\\ 0\le y\le0.5}}f(x,y)=y^2,$

$$\varphi1_1^{-}(y)=\lim_{\substack{x\to0.5-0\\ 0\le y\le0.5}}f(x,y)=y^2+0.25;$$

$$\varphi2_0^{+}(x)=\lim_{\substack{y\to+0\\ 0\le x\le0.5}}f(x,y)=x^2,$$

$$\varphi2_1^{-}(x)=\lim_{\substack{y\to0.5-0\\ 0\le x\le0.5}}f(x,y)=x^2+0.25;$$

$\Pi_{12}:\ \varphi1_0^{+}(y)=\lim\limits_{\substack{x\to+0\\ 0.5\le y\le1}}f(x,y)=y^2,$

$$\varphi1_1^{-}(y)=\lim_{\substack{x\to0.5-0\\ 0.5\le y\le1}}f(x,y)=y^2-0.25;$$

$$\varphi2_1^{+}(x)=\lim_{\substack{y\to0.5+0\\ 0\le x\le0.5}}f(x,y)=0.25-x^2,$$

$$\varphi2_1^{-}(x)=\lim_{\substack{y\to1-0\\ 0\le x\le0.5}}f(x,y)=1-x^2;$$

$\Pi_{21}:\ \varphi1_0^{+}(y)=\lim\limits_{\substack{x\to0.5+0\\ 0\le y\le0.5}}f(x,y)=y^2-0.25,$

$$\varphi1_1^{-}(y)=\lim_{\substack{x\to1-0\\ 0\le y\le0.5}}f(x,y)=y^2-1;$$

$$\varphi2_0^{+}(x)=\lim_{\substack{y\to+0\\ 0.5\le x\le1}}f(x,y)=-x^2,$$

$$\varphi2_1^{-}(x)=\lim_{\substack{y\to0.5-0\\ 0.5\le x\le1}}f(x,y)=0.25-x^2;$$

$\Pi_{22}:\ \varphi1_0^{+}(y)=\lim\limits_{\substack{x\to0.5+0\\ 0.5\le y\le1}}f(x,y)=-y^2-0.25,$

$$\varphi1_1^{-}(y)=\lim_{\substack{x\to1-0\\ 0.5\le y\le1}}f(x,y)=-y^2-1;$$

$$\varphi2_0^{+}(x)=\lim_{\substack{y\to0.5+0\\ 0.5\le x\le1}}f(x,y)=-x^2-0.25,$$

$$\varphi2_1^{-}(x)=\lim_{\substack{y\to1-0\\ 0.5\le x\le1}}f(x,y)=-x^2-1.$$

And a given matrix of interpolation data in the nodes of a given grid

$$C=\begin{pmatrix}0&0.25&0.25&0.5\\0.25&1&0&0.75\\-0.25&0&-1&-0.75\\-0.5&-1.25&-1.25&-2\end{pmatrix}.$$

Construct an interlination spline for a rectangular element Π_{11}.

$$S1_{11}(x,y)=\varphi1^{+}{}_0(y)\cdot\frac{x-x_1}{x_0-x_1}+\varphi1^{-}{}_1(y)\cdot\frac{x-x_0}{x_1-x_0}=$$
$$=y^2\cdot\frac{x-0.5}{-0.5}+(y^2+0.25)\cdot\frac{x-0}{0.5-0}=$$
$$=-2y^2(x-0.5)+2x(y^2+0.25);$$

$$S2_{11}(x,y)=\varphi2^{+}{}_0(x)\cdot\frac{y-y_1}{y_0-y_1}+\varphi2^{-}{}_1(x)\cdot\frac{y-y_0}{y_1-y_0}=$$
$$=x^2\cdot\frac{y-0.5}{0-0.5}+(0.25+x^2)\cdot\frac{y}{0.5}=$$
$$=-2x^2(y-0.5)+2(0.25+x^2)y;$$

$$S12_{11}(x,y)=C^{++}_{00}\frac{x-x_1}{x_0-x_1}\frac{y-y_1}{y_0-y_1}+C^{+-}_{01}\frac{x-x_1}{x_0-x_1}\frac{y-y_0}{y_1-y_0}+$$
$$+C^{-+}_{10}\frac{x-x_0}{x_1-x_0}\frac{y-y_1}{y_0-y_1}+C^{--}_{11}\frac{x-x_0}{x_1-x_0}\frac{y-y_0}{y_1-y_0}=$$
$$=0.25\frac{x-0.5}{-0.5}\frac{y}{0.5}+0.25\frac{x}{0.5}\frac{y-0.5}{-0.5}+0.5\frac{x}{0.5}\frac{y}{0.5}=$$
$$-y(x-0.5)-x(y-0.5)+2xy;$$

$$S_{11}(x,y)=S1_{11}(x,y)+S2_{11}(x,y)-S12_{11}(x,y)=$$
$$=-2y^2(x-0.5)+2x(y^2+0.25)-2x^2(y-0.5)+$$
$$+2(0.25+x^2)y=(x-0.5)+x(y-0.5)-2xy=y^2+x^2.$$

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Interlination splines are similarly constructed on other rectangular elements. As a result, we obtain a discontinuous interlination spline

$$S(x,y)=\begin{cases} x^2+y^2, & (x,y)\in\Pi_{11}, \\ x^2-y^2, & (x,y)\in\Pi_{12}, \\ -x^2+y^2, & (x,y)\in\Pi_{21}, \\ -x^2-y^2, & (x,y)\in\Pi_{22}. \end{cases}$$

5 RESULTS

This section provides a testing discontinuous approximation and interlination splines to recover a discontinuous function of two variables. Information about the function f (x, y) is given by the corresponding one-sided traces of a discontinuous function along a given system of lines for an interlining spline and one-sided values of the function at the nodes of a given perpendicular grid for a discontinuous approximation spline. We determine the maximum deviation of the approximate function $f(x,y)$ from the constructed bilinear spline $L(x,y)$

$$\max\left|f(x,y)-L(x,y)\right|\approx 0.064.$$

Based on the obtained form of discontinuous interlination spline $L(x,y)$, it completely coincides with the approximating function, i.e. $\max\left|f(x,y)-S(x,y)\right|=0.$

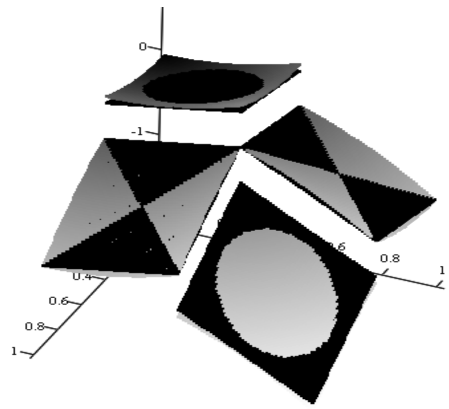

Figure 3 – Graphical view of the given function $f(x,y)$ (gray color) and the received spline $L(x,y)$ (black color)

6 DISCUSSION

We can conclude that the interlination discontinuous spline accurately restores a given discontinuous function on a given rectangular grid of nodes in contrast to the discontinuous approximation spline. This indicates that the numerical experiment confirms the theoretical results presented in this paper.

However, it should be noted that the discontinuous structures constructed in the article are used for experimental data of a different nature (values of the desired function at points and on lines).

CONCLUSIONS

The problem of development and research of methods for approximation of discontinuous functions by discontinuous interlination splines and its further application to problems of computed tomography are considered in this paper.

The scientific novelty is that for the first time a discontinuous interlining operator has been constructed to restore functions of two variables that have discontinuities of the first kind. The main difference from classical approximation methods is that discontinuous interlining uses one-sided traces as information functions along given lines. Constructed discontinuous structure high approximation accuracy.

The practical significance of this work is that new methods using information about a function in the form of one-sided traces of a function along a given system of lines opens up new ways in the construction of mathematical models, particularly in computed tomography.

Prospects for further research are to construct and study of a method for finding lines of discontinuity of a function of two variables in the case when information about the function is given in the form of traces of a function on a given system of lines and to develop a theory of discontinuous splines on areas of complex shape bounded by arcs of known curves.

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ВІДНОВЛЕННЯ РОЗРИВНОЇ ФУНКЦІЇ РОЗРИВНИМИ ІНТЕРЛІНАЦІЙНИМИ СПЛАЙНАМИ

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АНОТАЦІЯ

Актуальність. Проблема розробки та дослідження методів апроксимації розривних функцій розривними інтерлінаційними сплайнами та її подальше застосування до задач комп'ютерної томографії. Об'єктом дослідження є моделювання об'єктів з розривною внутрішньою структурою. Мета роботи – дослідження та розробка загального методу побудови розривних інтерлінаційних поліноміальних сплайнів, які, як окремий випадок, включають розривні та неперервно-диференційовані сплайни.

Метод. Сучасні методи відновлення функцій характеризуються новими підходами до отримання, обробки та аналізу інформації. Виникає потреба в побудові математичних моделей, в яких інформація може бути представлена не тільки значеннями функції в точках, а й у вигляді набору слідів функцій на площинах або прямих. Водночас практика показує, що серед багатовимірних об'єктів, які потребують дослідження, більше проблем описуються розривними функціями.

У статті розроблено загальний метод побудови розривних інтерлінаційних поліноміальних сплайнів, до складу яких, як окремий випадок, входять розривні та неперервно диференційовані сплайни. Вважається, що область визначення шуканої двовимірної функції розбита на прямокутні елементи. Сформульовано та доведено теореми про інтерлінаційні та апроксимаційні властивості таких розривних конструкцій. Розроблено метод апроксимації розривних функцій двох змінних на основі побудованих розривних сплайнів. Вхідними даними є сліди невідомої функції вздовж заданої системи взаємно перпендикулярних прямих. Запропонований метод має не тільки теоретичне значення, а й практичне застосування в сфері IT, особливо в комп'ютерній томографії, що дозволяє більш точно відновити внутрішню структуру організму.

Результати. Досліджено оператор розривної інтерлінації за відомими слідами функції двох змінних на системі взаємно перпендикулярних прямих.

Висновки. Функції двох змінних, які є розривними в деяких точках або на деяких лініях, краще апроксимуються розривними інтерлінаційними сплайнами. При цьому можна отримати однаково високі оцінки наближення. Отримані результати мають значні переваги перед існуючими методами інтерполяції та апроксимації розривних функцій. У подальших дослідженнях автори планують розвинути теорію розривних сплайнів на ділянках складної форми, обмежених дугами відомих кривих.

КЛЮЧОВІ СЛОВА: обробка сигналів, поліноміальний сплайн, інтерлінація, розривні функції, апроксимація.

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