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INNOVATIVE IMPROVED APPROXIMATE SOLUTION METHOD FOR THE INTEGER KNAPSACK PROBLEM, ERROR COMPRESSION AND COMPUTATIONAL EXPERIMENTS

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ABSTRACT

Context. Mathematical models of many optimization problems encountered in economics and engineering are taken in the form of an integer knapsack problem. Since this problem belongs to the class of "NP-complete", that is, "hard to solve" problems, the number of operations required by known methods to find its optimal solution is exponential. This does not allow solving large-scale problems in real time. Therefore, various and fast working approximate solution methods of this problem have been developed. However, it is known that the approximate solution provided by those methods can differ significantly from the optimal solution in most cases. Therefore, after taking any approximate solution as a starting point, there is a demand to develop methods for its further improvement. Development of such methods has both theoretical and great practical importance.

Objective. The main purpose solving of this issue is as follows**.** The main purpose in performing this work is to first find an initial approximate solution of the problem using any known method, and then work out an algorithm for successively further improvement of this solution. For this purpose, the set of numbers with which the coordinates of the optimal solution and the found approximate solution can differ should be determined. After that, new solutions should be constructed by assigning possible values to the unknowns corresponding to the numbers in that set, and the best among these solutions should be selected. However, the algorithm for constructing such a solution should be simple, require a small number of operations, not cause difficulties from the point of view of programming, be new and be applicable to practical issues.

Method. The essence of the proposed method consists of the following. First, the initial approximate solution of the considered problem and the value of the objective function corresponding to this solution are found by a known rule. After that, the optimal solution of the problem is easily found by a known method, without taking into account the condition that the unknowns are integers**.** Obviously, this solution can take at most one coordinate fractional value. It is assumed that the coordinates of the optimal solution of the integer knapsack problem and the initial approximate solution may differ around a certain fractional coordinate of the optimal solution of the continuous problem. Then, the minimum number of non-zero coordinates and zero coordinates in the optimal solution is found. Corresponding theorems have been proved for this. It is assumed that the different coordinates of the optimal solution and the initial approximate solution located between those minimal numbers. Therefore, the best solution can be selected by successively changing the coordinates between those minimum numbers one by one.

Results. Extensive calculation experiments were conducted with the application of the proposed method.To have a high quality of this method was confirmed once again through experiments.

Conclusions. The proposed method is new, simple in nature, easy to consider from the programming point of view, and has important practical importance. Thus, we call this solution the innovative improved approximate solution.

KEYWORDS: Integer knapsack problem, initial approximation solution, minimum number of zeros and non-zero coordinates in optimal solution, innovative improvement of initial approximation solution, error estimation and experiments.

NOMENCLATURE

N – number of issues resolved;

n – number of unknowns;

$a_j, c_j, d_j (j=\overline{1,n})$ and b are given positive integers;

$x_j, (j=\overline{1,n})$ – j-th unknown;

X^t – approximate solution;

f^t – value of the approximate solution to the objective function;

X^* – optimal solution of the problem;

f^* – optimal value of the objective function;

X^{lp} – optimal solution of the appropriate linear programming problem;

k – number of the fractional coordinate in the numerical x^{lp} solution;

f^{lp} – optimal value of the objective function in the appropriate linear programming problem;

x_k – coordinate , that fractional coordinate in the x^{lp} solution;

p, q, δ – mentioned positive integer;

δ_k^1, δ_k^0 – certain integers expressed as a percentage;

$n(1)$ and $n(0)$ – minimal number of non-zero coordinates and zeros, respectively, in the optimal solution of the considered problem;

ω_1, ω_2 – certain set of numbers, respectively;

$n(\omega_1), n(\omega_2)$ – number of elements of matching sets;

$\underline{X^*}$ and $\underline{X^{lp}}$ – optimal solutions of certain problems;

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n_1 and $\bar{n}$ – respectively, is the minimum number of non-zero coordinates and the maximum number of zero coordinates in the optimal solution of a certain problem;

X^{it} – innovative improved final solution;

f_i^t – value of the objective function according to the solution X^{it};

δ and δ^i respectively, are the relative errors (in percent) of the initial approximate solution and the innovative improved solution from the optimal solution.

INTRODUCTION

Consider the following integer knapsack problem:

$$\sum_{j=1}^{n} c_j x_j \to \max, \qquad (1)$$

$$\sum_{j=1}^{n} a_j x_j \le b, \qquad (2)$$

$$0 \le x_j \le d_j, \quad (j=\overline{1,n}) \qquad (3)$$

$$x_j - \ integer \ (j=\overline{1,n}). \qquad (4)$$

Here, without breaking generality, we accept that – $a_j > 0, c_j > 0, d_j > 0,\ (j=\overline{1,n})$ and $b > 0$ are given integers.

The problem (1.1)–(1.4) is called the integer knapsack problem or the one-constrained integer programming problem in the literature [1–3], etc.

Note that, Since the problem (1)–(4) belongs to the NP-complete class, that is, to the class of "hard-to-solve problems", the maximum number of operations required by known methods (branches and bounds, dynamic programming and some combinatorial type) to find its optimal solution is exponential is from the compilation. Therefore, it is not possible to solve large-scale problems with these methods in real time. Therefore, certain approximate solution methods of problem (1.1)–(1.4) have been developed [2, 3, 6, 7], etc. These methods mainly based on the criterion

$$\max_j \frac{c_j}{a_j} = \frac{c_{j*}}{a_{j*}}. \qquad (5)$$

So, for the number j_* found from relation (5), the coordinate x_{j_*} is given the maximum value that satisfies the conditions (2)–(4). Then, the next new number j_* is found from relation (5) and the corresponding coordinate x_{j_*} is given a value.

The process of constructing such a solution ends after evaluating all n number of coordinates. In this case, the value is found by looking at each coordinate only once, and the number of operations required is at most $O(n^2)$ compilation.

Without violating generality, let us assume that the coefficients of problem (1)–(4) satisfy the following relations:

$$\frac{c_1}{a_1} \ge \frac{c_2}{a_2} \ge \ldots \ge \frac{c_k}{a_k} \ge \ldots \ge \frac{c_n}{a_n}. \qquad (6)$$

Then, if we ignore the completeness condition on the variables x_j, $(j=\overline{1,n})$ in the considered problem, the received continuous problem (1)–(3) turns into a simple linear programming problem. The optimal solution of this obtained problem is easily found analytically, and only one coordinate may not be an integer in this solution.

Suppose, this is k-th coordinate and is like $x_k = \frac{\alpha}{\beta}$.

We assume that the coordinates of the optimal solution of the integer knapsack problem (1)–(4) and the coordinates of the initial approximate solution differ around a certain k-th coordinate. Note that, we came to this conclusion after numerous numerical experiments, and the same result was also given in [8, 13]. If we can find those different coordinates and give a new value, it is natural that a better solution can be obtained. Such a solution algorithm is proposed in this work. For this purpose, a suitable theorem for finding the minimum number of coordinates with "0" values and values different from "0" in the optimal solution of problem (1)–(4) has been proved. In this time, we assume that the initial minimal number of non-zero coordinates and the minimal number of zeros in the easily found optimal solution of the continuous problem (1)–(3) also coincide with the optimal solution of the integer knapsack problem (1)–(4). Therefore, the initial approximate solution may have coordinates that differ from the optimal solution in that range. Because, in this range, c_j / a_j ratios are not significantly different from each other. Therefore, we can get a better solution by changing the coordinates in this range within the conditions (2)–(4).

Note that such an idea was used in works [4, 5].

1 PROBLEM STATEMENT

Suppose we consider problem (1)–(4) and here relations (6) are satisfied. Then the approximate solution of that problem $X^t = (x_1^t, x_2^t, \ldots, x_n^t)$ is found by the following formula:

$$x_j^t = \begin{cases} d_j, \text{if } \ a_j d_j \le b - \sum_{i=1}^{j-1} a_i x_i^t, \\ \left[\left(b - \sum_{i=1}^{j-1} a_i x_i^t\right) / a_j\right], \text{if } \ a_j d_j > b - \sum_{i=1}^{j-1} a_i x_i^t. \end{cases} \qquad (7)$$

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Here, j=1,2,…,n takes values and [z] is the integer part of the number z. Substituting this solution in the function (1), we get the approximate

$$f^t = \sum_{j=1}^{n} c_j x_j^t$$

value of that problem.

If the optimal solution of the problem (1)–(4) is $X^* = (x_1^*, x_2^*, ..., x_n^*)$, then the maximum f^* value of the function (1) in this problem is as follows

$$f^* = \sum_{j=1}^{n} c_j x_j^*.$$

Obviously it must be $f^t \le f^*$. However, since finding the optimal solution $X^* = (x_1^*, x_2^*, ..., x_n^*)$ is related to serious difficulties, it is necessary to find the upper limit of the approximate value of f^*. Because the approximation of the approximate solution to the optimal solution should be evaluated. For this purpose, we need to solve a simple linear programming problem, ignoring the completeness condition (4) in problem (1)–(4). We call that issue an unbreakable issue. Because here the condition of being integers is not imposed on the unknowns. Note that the optimal solution of obtained unbreakable issue (1)–(3), i.e. simple linear programming problem $X^{lp} = (x_1^{lp}, x_2^{lp}, ..., x_n^{lp})$ is found by the following well-known formula.

$$x_i^{lp} = \begin{cases} d_j, \text{if } a_j d_j \le b - \sum_{i=1}^{j-1} a_i x_i^{lp}, \\ (b - \sum_{i=1}^{j-1} a_i x_i^{lp}) / a_j, \text{if } a_j d_j > b - \sum_{i=1}^{j-1} a_i x_i^{lp}, (k := j), \\ 0, \text{ if } j = k+1, k+2, ..., n. \end{cases} \quad (8)$$

So, in this solution, only the k-th x_k^{lp} variable can take a fractional value. It is known that, if the coordinate x_k is an integer, then this solution is the optimal solution of problem (1)–(4). By substituting the solution of (8) into the function (1) we will get the number

$$f^{lp} = \sum_{j=1}^{n} c_j x_j^{lp}.$$

İt is clear that, the number f^{lp} is the upper limit of the approximate number f^t. Because we do not take into account the condition of (4) being integers over the unknowns, the set of possible solutions of the problem increases, and the maximum value obtained at this time will also be a large value. So, the following relation is true:

$$f^t \le f^* \le [f^{lp}].$$

Here, the symbol $[f^{lp}]$ indicates the integer part of the number f^{lp}. Thus, if we denote $[f^{lp}] = \overline{f}$, this number will be the upper limit of the approximate value of f^t. As a result we get $f^t \le f^* \le \overline{f}$. This relationship allow us that, to estimate the approximation of the found f^t approximation to the optimal solution, that is, the absolute or relative error.

Numerous experiments and the solution of real practical problems show that the approximate solution of (7) found by the known classical way and the corresponding approximate value of f^t can differ significantly from the optimal solution X^* and the corresponding f^* number. Therefore, our goal in performing this presented work is to develop an algorithm to find a better solution than the solution $X^t = (x_1^t, x_2^t, ..., x_n^t)$. However, this algorithm should be simple, give a greater value to the function (1) than the initial solution, should not be difficult from the programming point of view, and should be suitable for solving real practical problems. Therefore, we will call such a solution found an innovative improved approximate (suboptimal) solution.

2 REVIEW OF THE LITERATURE

First of all, let us note that the integer knapsack problem, including the Boolean programming problems, have been known since the last century. Since these issues have wide practical applications, various solution methods have been developed to find their optimal solutions [1–3]. Since these issues have wide practical applications, various solution methods have been developed to find their optimal solutions [1–3]. But it soon became clear that none of those methods had polynomial time complexity. In other words, the Boolean programming problem, as well as the integer bag problem, belong to the NP-complete class, that is, to the class of hard-to-solve problems [4]. Therefore, various approximate (suboptimal) solving methods were developed for this class of problems [2, 3, 5–6, etc]. On the other hand, taking into account that these issues are of wide practical importance, their more generalized models began to be applied [7–10, 14, etc.]. Here, generalization means that the given coefficients are located in certain intervals. In some works, the methods of finding the stability interval of the optimal solution and finding the generalized solution in certain problems whose coefficients are intervals have been developed [11–12 etc.].

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Despite all this, certain studies have been conducted to further improve the approximate solutions found in integer programming problems [8, 9, 13, etc.]. In the article we have presented, the problem of finding an initial approximate solution to the integer knapsack problem has been considered.

In [15, 16, 20–23], certain approximate solution methods of knapsack problems described by various models were developed. Certain methods of solving the knapsack problem, whose initial data are in the form of intervals, are given in works [17–19]. A certain relationship between the optimal solution of the linear programming problem and the approximate solution of the integer programming problem was considered in [24], and the average case analysis of solutions of the knapsack problem with greedy algorithms was considered in [25].

3 MATERIALS AND METHODS

First, let's note that we can write formulas (7) and (8) more concisely as follows. Because such writing is more convenient from the point of view of programming. For each $j, (j = 1, 2, ..., n)$ numbers

$$x_j^t = \min\left\{d_j, \left[(b - \sum_{i=1}^{j-1} a_i x_i^t)/a_j\right]\right\}, \quad (9)$$

$$x_j^{lp} = \min\left\{d_j, (b - \sum_{i=1}^{j-1} a_i x_i^t)/a_j\right\}. \quad (10)$$

Note that when finding the solution of the continuous problem (1)–(3) $X^{lp} = (x_1^{lp}, x_2^{lp}, ..., x_n^{lp})$ with the formula (10) for a certain first number $j=k$

$$(b - \sum_{i=1}^{j-1} a_i d_i)/a_j \le d_j,$$

then we remember that number k and it is clear that $x_j^{lp} = 0$ for the numbers $j = k+1, k+2, ..., n$.

Thus, the optimal solution of the continuous problem (1)–(3) with the formula (10) is in the following form:

$$X^{lp} = (d_1, d_2, ..., d_{k-1}, \frac{\alpha}{\beta}, 0, ..., 0). \quad (11)$$

If the number $x_k = \frac{\alpha}{\beta}$ in the solution (11) is an integer, then this solution is the optimal solution of the integer knapsack problem (1)–(4) and no further research is needed.

Assume $x_k = \frac{\alpha}{\beta}$ is a fractional number.

Conducted numerous experiments show that the coordinates of the optimal solution $X^* = (x_1^*, x_2^*, ..., x_n^*)$ of the problem (1)–(4) are approximately $X^t = (x_1^t, x_2^t, ..., x_n^t)$ solution coordinates differ only around the k-th x_k^{lp} coordinate in formula (11). Because, around certain $j \in [k-p; k+q]$ of the x_k^{lp} coordinate, $\frac{c_j}{a_j}$ ratios do not differ significantly from each other. The selection of numbers p and q will be reported below.

Note that according to formula (11) $x_j^{lp} = 0$ for numbers $j = k+1, k+2, ..., k+q$ and $x_j^{lp} \ne 0$ for numbers $j = k-p, k-p+1, ..., k$. Therefore, in order to get a better solution, we can construct the new solution X^t with the formula (7) by successively writing $x_j = d_j - 1, d_j - 2, ..., 0$ for each number $j = k+1, k+2, ..., k+q$ to the right of number k and for number $j = k-p, k-p+1, ..., k$ to the left of number k. At this time, for each approximate solution of X^t we calculate the new f^t value of the function (1) and remember the largest value and the corresponding X^t solution. We called this last-mentioned solution an innovative improved solution. Obviously, this solution will not be worse than the original solution found by the known formula (7).

It should be noted that the number p used in the interval $[k-p; k+q]$ in this article is the minimal $n(1)$ number of non-zero coordinates in the optimal solution of the problem (1)–(4) and the number q is found through the minimum $n(0)$ number of zeros in that solution. More precisely, it is chosen as $p = k - n(1), q = n - n(0) - k$. We will give the procedure for finding or evaluating the numbers $n(1)$ and $n(0)$ below.

It should be remembered that the issue of finding a better solution by choosing a certain neighborhood of the k-th coordinate in the formula (11) and finding a better solution was discussed in [8,9,13]. In [13], the number p was chosen as follows to determine the neighborhood of the k-th coordinate $[k-p, k+p]$, which received a fractional value.

$$p = \arg\left\{\max_i \frac{c_k - i}{a_k - i} - \frac{c_k + i}{a_k + i} \le \delta\right\}.$$

Here δ is the positive integer specified previously. As it can be seen, if the number k is close to the last coordinates or the first coordinate of the solution (11), then it may not be possible to select the symmetric interval $[k-p, k+p]$. In the works of [8,9], only the knapsack problem with Bul variable was considered and

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the neighborhood of the k-th coordinate $[\delta_k^1, \delta_k^0]$ was selected there.

Thus, $\delta_k^1 = [k \cdot \frac{q}{100}]$, $\delta_k^0 = [(n-k) \cdot \frac{q}{100}]$ are defined, and the number q is the minimum number of ones or zeros in the optimal solution of the continuous knapsack problem. Here, too, the selection of the number q can cause some misunderstandings. So, when the number q indicates the number of units, the interval δ_k^1, δ_k^0 differs from the interval obtained when the number q indicates the number of zeros.

In this work, the neighborhood of the coordinate $x_k = \frac{\alpha}{\beta}$ in the solution of (11) is chosen as $[k-p; k+q] = [n(1), n-n(0)]$.

As you can see, the relation $n(1) \le k \le n - n(0)$ is fulfilled. On the other hand, in order not to look at all the numbers located in this interval, in other words, to get the same result with a small number of operations, we need to find the minimal number $n(1)$ of non-zero coordinates and the minimal number $n(0)$ of zeros in the optimal solution of problem (1)–(4)..In this case, in the process of constructing a better solution, we should not change the first $n(1)$ number of non-zero coordinates and the last $n(0)$ number of zeros in the solution (11).

For this purpose, let's look at the following issues:

$$\sum_{j=1}^{n} x_j \to \min, \tag{12}$$

$$\sum_{j=1}^{n} a_j x_j \le b, \tag{13}$$

$$\sum_{j=1}^{n} c_j x_j \ge f^t, \tag{14}$$

$$0 \le x_j \le d_j, \ (j = \overline{1,n}), \tag{15}$$

$$x_j - \ integer \ \ (j = \overline{1,n}) \tag{16}$$

and

$$\sum_{j=1}^{n} x_j \to \max, \tag{17}$$

$$\sum_{j=1}^{n} a_j x_j \le b, \tag{18}$$

$$\sum_{j=1}^{n} c_j x_j \ge f^t, \tag{19}$$

$$0 \le x_j \le d_j, \quad (j = \overline{1,n}), \tag{20}$$

$$x_j - \ integer \quad (j = \overline{1,n}). \tag{21}$$

Note that by solving problem (12)–(16), we can find the minimum n(1) number of non-zero coordinates in the optimal solution of problem (1)–(4) as number $n(\omega_1)$ is the number of elements of the set ω_1.

Thus, $\omega_1 = \{j \mid \underline{x}_j^* > 0\}$, and $\underline{X}^* = (\underline{x}_1^*, \underline{x}_2^*, ..., \underline{x}_n^*)$ is the optimal solution of problem (12)–(16).

It is important to note that, the problem (12)–(16) is the special constrained integer programming problem. Obviously that, this problem from NP- complete class too and it is not easy to find their optimal solution $\underline{X}^*$. Therefore, if we do not take into account the completeness condition (16) imposed on the unknowns in that problem, the range of possible solutions of this problem will expand. Therefore, by solving the obtained linear programming problem (12)–(15), can be found the solution $\underline{X}^{lp} = (\underline{x}_1^{lp}, \underline{x}_2^{lp}, ..., \underline{x}_n^{lp})$. Then the number n (1) of non-zero coordinates in this solution is found as follows:

$$\underline{n}(1) = n(\omega_2).$$

Here is $\omega_2 = \{j \mid \underline{x}_j^{lp} > 0\}$.

It is clear that, it should be $\underline{n}(1) \le n(1)$. Because, when the area grows, the minimum price can decrease. However, since finding the number $\underline{n}(1)$ is related to the solution of the linear programming problem (12)–(15) with mixed constraints, we can still encounter certain difficulties. Therefore, instead of problem (12)–(16), can be considered the following problem with a larger domain.

$$\sum_{j=1}^{n} x_j \to \min,$$

$$\sum_{j=1}^{n} c_j x_j \ge f^t,$$

$$0 \le x_j \le d_j, \ (j = \overline{1,n}).$$

In this problem, without breaking generality, let us assume that the relation $c_1 \ge c_2 \ge ... \ge c_{n-1} \ge c_n$ is satisfied. Then the minimal number of non-zero coordinates n_1 in the optimal solution of this problem can be found as follows:

$$\sum_{j=1}^{n_1} c_j d_j \le f^t \le \sum_{j=1}^{n_1+1} c_j d_j .$$

It is clear that, the relationship $n_1 \le \underline{n}(1) \le n(1)$ is satisfied.

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Thus, we can take the number n_1 which is easily found, as the number of coordinates different from zero in the optimal solution of the problem (1)–(4).

Now let's find the minimum number of zeros in the optimal solution of problem (1)–(4). For this purpose, by making judgments according to the above, we have to solve the problem (17)–(21) or the problem (17)–(20), and finally the problem (22)–(25):

$$\sum_{j=1}^{n} x_j \to \max, \quad (22)$$

$$\sum_{j=1}^{n} a_j x_j \le b, \quad (23)$$

$$0 \le x_j \le d_j,\ (j=\overline{1,n}), \quad (24)$$

$$x_j - \text{integer} \quad (j=\overline{1,n}). \quad (25)$$

In the case of (22)–(25), let us assume, without violating the generality, that the relations $a_1 \le a_2 \le ... \le a_n$ are fulfilled in the condition (23). Then the maximal number $\bar{n}$ of non-zero coordinates in the optimal solution of this problem is found from the following relationship:

$$\sum_{j=1}^{\bar{n}} a_j d_j \le b \le \sum_{j=1}^{\bar{n}+1} a_j d_j.$$

It is clear that, we can take the number $\underline{n}(0) = n - \bar{n}$ instead of the minimal $n(0)$ number of zeros in the optimal solution of problem (1)–(4).

Thus, we proved the following theorem.

Theorem: The inequalities $\underline{n}(1) \le n(1)$ and $\underline{n}(0) \le n(0)$ for the minimum $n(1)$ number of non-zero coordinates and the minimum $n(0)$ number of zeros in the optimal solution of problem (1)–(4) it is true.

Let's note that the coordinates of the optimal solution of the problem (1)–(4) with the solution (11) can differ in the interval $[\,\underline{n}(1), n - \underline{n}(0)\,]$.

It is clear that d_j numbers are to the left and only zeros are to the right of the k-th coordinate in the solution of (3.3). Therefore, for each number j, ($j=\underline{n}(1), \underline{n}(1)+1,...,k$) to the left of the $x_j = 1,2,...,d_j$, $x_j = d_{j-1}, d_{j-2},...,0$ and $x_j = 1,2,...,d_j$ for each ($j=k+1$, $k+2,...$) numbers on the right, new solutions can be constructed by formulas (7) or (9). We will select the best of those solutions and consider it as an innovative improved approximate solution.

Now, let's write the algorithm for the innovative approximate solution method described above.

ALGORITHM

Step 1. Enter the numbers $n, b, c_j, a_j, d_j\ (j=\overline{1,n})$ and accept $bb := b$;

Step 2. For each number $j, (j=1,2,...,n)$ you need to find the solution $X^{lp} = (x_1^{lp}, x_2^{lp},..., x_k^{lp},..., x_n^{lp})$ with the formula

$$x_j^{lp} = \begin{cases} d_j, \text{ if } a_j d_j \le b - \sum_{i=1}^{j-1} a_i x_i^{lp}, \\ (b - \sum_{i=1}^{j-1} a_i x_i^{lp}) / a_j, \quad \text{if} \quad a_j d_j > b - \sum_{i=1}^{j-1} a_i x_i^{lp},\ (k := j), \\ 0, \quad \text{when} \quad j = k+1,\ k+2,...,n \end{cases}$$

and from here the fractional coordinate number k should be noted. kk:=k; r:=0;

Step 3. If x_k^{lp} is an integer, then the solution X^{lp} is the optimal solution of problem (1)–(4). At this time we should calculate

$$f^* := \sum_{j=1}^{n} c_j x_j^{lp},$$

$X^* = (x_1^*, x_2^*,..., x_n^*) = (x_1^{lp}, x_2^{lp},..., x_n^{lp})$, print f^* and go to step 21.

Step 4. To find the approximate solution of $X^t = (x_1^t, x_2^t,..., x_n^t)$ we need to calculate

$$x_j^t = \begin{cases} d_j, \text{ if } a_j d_j \le b - \sum_{i=1}^{j-1} a_i x_i^t, \\ [(b - \sum_{i=1}^{j-1} a_i x_i^t) / a_j], \quad \text{if} \quad a_j d_j > b - \sum_{i=1}^{j-1} a_i x_i^t, \end{cases}$$

for each number $(j=1,2,...,n)$.

Step 5. Calculate the numbers

$$f^{lp} = \sum_{j=1}^{n} c_j x_j^{lp},\ f^t = \sum_{j=1}^{n} c_j x_j^t.$$

Accept $f^{it} := f^t$ and remembered the solution $X^{it} = (x_1^t, x_2^t,..., x_n^t)$ with f^{it}.

Step 6. Coefficients $c_j, (j=\overline{1,n})$ should be arranged as $c_1 \ge c_2 \ge ... \ge c_n$ and to find the minimal n_1 number of non-zero coordinates in the optimal solution from the relationship

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$$\sum_{j=1}^{n_1} c_j d_j \le f^t \le \sum_{j=1}^{n_1+1} c_j d_j .$$

Step 7. Numbers $a_j, (j=\overline{1,n})$ should be arranged as $a_1 \le a_j \le ... \le a_n$ and find the maximum $\overline{n}$ number of coordinates different from zero in the optimal solution of the problem (1)–(4) from the relationship and note $n(0) = n - \overline{n}$

$$\sum_{j=1}^{\overline{n}_1} a_j d_j \le b \le \sum_{j=1}^{\overline{n}_1+1} a_j d_j$$

and note $\underline{n}(0) = n - \overline{n}$.

Step 8. Set $x_k^t := [x_k^t]$; $b := bb - a_k \times x_k^t$.

Step 9. For the numbers j, $j = 1,2,...,n$ $j \ne k$

$$x_j^t = \begin{cases} d_j, \; if \; a_j d_j \le b - \sum_{i=1}^{j-1} a_i x_i^t, \\ [(b - \sum_{i=1}^{j-1} a_i x_i^t)/a_j], \; if \; a_j d_j > b - \sum_{i=1}^{j-1} a_i x_i^t. \end{cases}$$

Step 10. Calculate

$$f^t := \sum_{j=1}^{n} c_j x_j^t .$$

If $f^t > f^{it}$, then should be accept $f^{it} := f^t$, $X^{it} = (x_1^t, x_2^t, ..., x_n^t)$.

Step 11. If r=0, then should be accept $x_k^t := [x_k^t] + 1$; $b := bb - a_k \times x_k^t$; $r := 1$ and go to step 9.

Step 12 Set be accept $r := 0$

Step 13. For each $j, (j = 1,2,...,n; j \ne k)$

$$x_j^t = \begin{cases} d_j, \; if \; a_j d_j \le b - \sum_{i=1}^{j-1} a_i x_i^t, \\ [(b - \sum_{i=1}^{j-1} a_i x_i^t)/a_j], \; if \; a_j d_j > b - \sum_{i=1}^{j-1} a_i x_i^t. \end{cases}$$

With the formula $X^t = (x_1^t, x_2^t, ..., x_n^t)$ calculate the number f^t

$$f^t := \sum_{j=1}^{n} c_j x_j^t .$$

Step 14. If $f^t > f^{it}$, ther $f^{it} := f^t$, $X^{it} = (x_1^t, x_2^t, ..., x_n^t)$ should be written and memorized. If $r := 1$ go to Step 18.

Step 15. If $x_k^t < d_k$, then go to Step 17.

Step 16. $k := k+1$; If $k > n(0)$ $k := kk$ and go to the Step 18.

Step 17. $x_k^t := 1,2,...,d_k$ and accordingly by taking $b := bb - a_k \times x_k^t$ go to the Step 13.

Step 18. $r := 1$; If $x_k^t = d_k$, then $k := k-1$; If $k < n(1)$ go to the Step 20.

Step 19. $x_k^t := 1,2,...,d_k$ k values corresponding to $b := bb - a_k \times x_k^t$ and go to Step 13.

Step 20. Print f^{it}, $X^{it} = (x_1^{it}, x_2^{it}, ..., x_n^{it})$. $\delta = ([f^{lp}] - f^t / [f^{lp}]$ and $\delta^i = ([f^{lp}] - f^{it} / [f^{lp}]$.

Step 21. STOP.

4 EXPERIMENTS

Numerous computational experiments have been conducted to investigate the quality of the innovative improved solution method we proposed above.

During the experiments, problems with a different number of variables were solved (n=100, n=300, n=500, n=1000). The coefficients of these problems were chosen as random numbers with two digits and three digits at most.

Thus, $0 < c_j \le 99$, $0 < a_j \le 99$ $d_j = 10, (j = \overline{1,n})$ or $0 < c_j \le 999$, $0 < a_j \le 999$ $d_j = 10, (j = \overline{1,n})$,

$$b = [\frac{1}{3} \sum_{j=1}^{n} a_j d_j].$$

It should be noted that 4 different problems, each with the same number of variables, were solved.

The results are given in the following tables and notations are adopted as follows.

N – the number of the solved problem with the same number of unknowns.

$\overline{f}$ – the upper bound of the optimal value of the problem (1.1)–(1.4).

f^t – the value of the function (1.1) according to the approximate solution of the problem (1.1)–(1.4) found by the known classical method.

f^{it} – the value given to the function (1.1) of the improved approximate solution.

δ – the relative error of the approximate value found by the classical method, expressed as a percentage of the optimal value. It mean that,

$$\delta = \frac{\overline{f} - f^t}{\overline{f}} \cdot 100 ,$$

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δ^i – an innovative improved approximate persentage is relative error.

$$\delta^i = \frac{\overline{f} - f^i}{\overline{f}} \cdot 100 .$$

5 RESULTS

Table 1 – Problems with two-digit coefficients ($n = 100$)

N	1	2	3	4
$\overline{f}$	34211.00	34493.00	37378.00	33508.00
f^t	34205.00	34486.00	37353.00	33492.00
f^{it}	34209.00	34491.00	37376.00	33506.00
δ	0.01754	0.02029	0.06688	0.04775
δ^i	0.00585	0.00580	0.00535	0.00597

Table 2 – Problems with two-digit coefficients ($n = 300$)

N	1	2	3	4
$\overline{f}$	103013.00	98887.00	97245.00	99692.00
f^t	103010.00	98884.00	97244.00	99684.00
f^{it}	103012.00	98885.00	97246.00	99693.00
δ	0.00291	0.00303	0.00103	0.00802
δ^i	0.00097	0.00202	0.00103	0.00100

Table 3 – Problems with two-digit coefficients (n=500)

N	1	2	3	4
$\overline{f}$	161131.00	171036.00	170555.00	168122.00
f^t	161128.00	171033.00	170555.00	168115.00
f^{it}	161130.00	171034.00	170555.00	168122.00
δ	0.00186	0.00175	0.00000	0.00416
δ^i	0.00062	0.00117	0.00000	0.00000

Table 4 – Problems with two-digit coefficients (n=1000)

N	1	2	3	4
$\overline{f}$	339009.00	330007.00	329556.00	335835.00
f^t	339009.00	330003.00	329555.00	335832.00
f^{it}	339009.00	330007.00	329556.00	335834.00
δ	0.00000	0.00121	0.00030	0.00089
δ^i	0.00000	0.00000	0.00000	0.00030

Table 5 – Problems with three-digit coefficients ($n = 100$)

N	1	2	3	4
$\overline{f}$	343742.	346062.00	375258.0	336331.00
f^t	343485.00	345998.00	375018.00	336155.00
f^{it}	343660.00	346040.00	375227.00	336305.00
δ	0.07477	0.01849	0.06396	0.05233
δ^i	0.02386	0.00636	0.00826	0.00773

Table 6 – Problems with three-digit coefficients ($n = 300$)

N	1	2	3	4
$\overline{f}$	1035261.00	993336.00	976972.00	1002027.00
f^t	1035236.00	993265.00	976948.00	1001939.00
f^{it}	1035251.00	993307.00	976948.00	10019 95.00
δ	0.00241	0.00715	0.00246	0.00878
δ^i	0.00097	0.00292	0.00246	0.00319

Table 7 – Problems with three-digit coefficients (n=500)

N	1	2	3	4
$\overline{f}$	1620326.00	1718600.00	1712542.00	1689749.00
f^t	1620292.00	1718561.00	1712502.00	1689670.00
f^{it}	1620303.00	1718587.00	1712528.00	1689739.00
δ	0.00210	0.00227	0.00234	0.00468
δ^i	0.00142	0.00076	0.00082	0.00059

6 DISCUSSION

Based on the tables, the following conclusions can be drawn.

In most cases, the initially found approximate solution has been further improved. Rather, in 24 out of 28 solved problems, the initial approximate solution was further improved. In the remaining 4 problems, the initial approximate solution has not improved. It can be assumed that this solution is the optimal solution. The relative errors of the found approximate values from the optimal value are very small and do not exceed 1%. This is very important for solving real practical problems. It should be noted that the algorithm proposed in the article does not count options, so it takes seconds to solve problems. Therefore, we did not mention the computer time in the tables.

CONCLUSIONS

A new approximate solution method of the integer knapsack problem is given in the presented article. Through this method, any initial solution found by known methods is successively improved. At this time, the

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interval where the coordinates that do not coincide with the optimal solution are located is determined. After that, it is possible to build a better solution by assigning new values to the coordinates in those intervals.

The proposed method is new, simple in nature, easy to consider from the programming point of view, and has important practical importance, so this method is called an innovative improved approximate solution method.

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ІННОВАЦІЙНИЙ ВДОСКОНАЛЕНИЙ МЕТОД НАБЛИЖЕНОГО РІШЕННЯ ДЛЯ ЗАДАЧІ ЦІЛОЧИСЕЛЬНОГО РАНЦЯ, СТИСНЕННЯ ПОМИЛОК ТА ОБЧИСЛЮВАЛЬНІ ЕКСПЕРИМЕНТИ

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АНОТАЦІЯ

Актуальність. Математичні моделі багатьох задач оптимізації, що зустрічаються в економіці та техніці, розглядаються у формі задачі про цілочисельний рюкзак. Оскільки ця задача належить до класу «NP-повних», тобто «важко розв'язуваних», кількість операцій, необхідних відомим методам для знаходження її оптимального розв'язку, експоненціальна. Це не дозволяє вирішувати масштабні завдання в режимі реального часу. Тому розроблено різноманітні та швидкопрацюючі методи наближеного розв'язання цієї задачі. Однак відомо, що наближене рішення, отримане цими методами, у більшості випадків може суттєво відрізнятися від оптимального. Тому після прийняття будь-якого наближеного рішення за вихідну точку виникає потреба розробити методи його подальшого вдосконалення. Розробка таких методів має як теоретичне, так і велике практичне значення.

Мета роботи. Основна мета вирішення цього питання полягає в наступному. Основна мета виконання даної роботи полягає в тому, щоб будь-яким відомим методом спочатку знайти вихідний наближений розв'язок задачі, а потім розробити алгоритм для послідовного подальшого вдосконалення цього розв'язку. Для цього необхідно визначити набір чисел, якими можуть відрізнятися координати оптимального і знайденого наближеного розв'язку. Після цього слід побудувати нові розв'язки шляхом присвоєння можливих значень невідомим, що відповідають числам цього набору, і вибрати найкраще з цих розв'язків. Але алгоритм побудови такого рішення повинен бути простим, вимагати невеликої кількості операцій, не викликати труднощів з точки зору програмування, бути новим і застосовним до практичних завдань.

Метод. Суть запропонованого способу полягає в наступному. Спочатку за відомим правилом знаходять початковий наближений розв'язок задачі, що розглядається, і відповідне йому значення цільової функції. Після цього оптимальний розв'язок задачі легко знаходить відомим методом без урахування умови цілості невідомих. Очевидно, що цей розв'язок може приймати не більше одного дробового значення координати. Передбачається, що координати оптимального розв'язку цілочисельної задачі про ранець і початкового наближеного розв'язку можуть відрізнятися навколо певної дробової координати оптимального розв'язку неперервної задачі. Потім знайдено мінімальну кількість ненульових координат і нульових координат в оптимальному розв'язку. Для цього доведено відповідні теореми. Передбачається, що різні координати оптимального розв'язку та початкового наближеного розв'язку знаходяться між цими мінімальними числами. Таким чином, найкраще рішення можна вибрати шляхом послідовної зміни координат між цими мінімальними числами один за одним.

Результати. Із застосуванням запропонованого методу були проведені численні розрахункові експерименти. Висока якість цього методу ще раз підтверджена експериментально.

Висновки. Запропонований метод є новим, простим за своєю суттю, легким для програмування та має важливе практичне значення. Таким чином, ми називаємо це рішення інноваційним покращеним наближеним рішенням.

КЛЮЧОВІ СЛОВА: задача про цілочисельний ранець, розв'язок початкового наближення, мінімальна кількість нульових і ненульових координат в оптимальному розв'язку, інноваційне вдосконалення розв'язку початкового наближення, оцінка похибки та експерименти.

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