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MATHEMATICAL FOUNDATIONS OF METHODS FOR SOLVING CONTINUOUS PROBLEMS OF OPTIMAL MULTIPLEX PARTITIONING OF SETS

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ABSTRACT

Context. The research object is the process of placing service centers (e.g., social protection services, emergency supply storage) and allocating demand for services continuously distributed across a given area. Mathematical models and optimization methods for location-allocation problems are presented, considering the overlap of service zones to address cases when the nearest center cannot provide the required service. The relevance of the study stems from the need to solve problems related to territorial distribution of logistics system facilities, early planning of preventive measures in potential areas of technological disasters, organizing evacuation processes, or providing primary humanitarian assistance to populations in emergencies.

Objective. The rational organization of a network of service centers to ensure the provision of guaranteed service in the shortest possible time by assigning clients to multiple nearest centers and developing the corresponding mathematical and software support.

Method. The concept of a characteristic vector-function of a *k*-th order partition of a continuous set is introduced. Theoretical justification is provided for using the LP-relaxation procedure to solve the problem, formulated in terms of such characteristic functions. The mathematical framework is developed using elements of functional analysis, duality theory, and nonsmooth optimization.

Results. A mathematical model of optimal territorial zoning with center placement, subject to capacity constraints, is presented and studied as a continuous problem of optimal multiplex partitioning of sets. Unlike existing models, this approach describes distribution processes in logistics systems by minimizing the distance to several nearest centers while considering their capacities. Several propositions and theorems regarding the properties of the functional and the set of admissible solutions are proven. Necessary and sufficient optimality conditions are derived, forming the basis for methods of optimal multiplex partitioning of sets.

Conclusions. Theoretical findings and computational experiment results presented in the study confirm the validity of the developed mathematical framework, which can be readily applied to special cases of the problem. The proven propositions and theorems underpin computational methods for optimal territorial zoning with center placement. These methods are recommended for logistics systems to organize the distribution of material flows while assessing the capacity of centers and the fleet of transportation vehicles involved.

KEYWORDS: continuous set, multiplex partitioning, optimization, LP-relaxation, optimality conditions, location-allocation problems.

ABBREVIATIONS

OMPS is an optimal multiplex partitioning of sets;

LP is Linear Programming.

NOMENCLATURE

Ω is a set being partitioned is bounded, closed, and Lebesgue measurable in the space E_2;

τ_i are points from Ω, which are called centers;

N is a number of centers;

k is an order of the partition of the set Ω;

N is a set of all center indices;

M is a set of all k-element subsets of the set **N**;

L is a number of all k-element subsets of the set **N**;

$\rho()$ is a non-negative function describing the demand for the service.

$c()$ is a cost function;

w_i is a proportionality coefficient;

a_i is a cost of establishing i-th center τ_i or upgrading an existing one, or its fixed operating costs calculated per unit of demand;

b_i is a capacity of i-th center, which defines the maximum volume of services the center can provide;

Ω_σ are areas covering clients who have the same k nearest neighboring service centers from N existing (possible);

σ is a set of indices of centers associated with the subset Ω_σ.

INTRODUCTION

The problems of efficient organization of logistics, production, and trade networks constitute one of the directions of modern optimization theory. The scientific literature contains numerous works devoted to the location-allocation problem – determining the best positions for service centers along with the most rational distribution of demand for the services they provide. A comprehensive review of location-allocation models and methods, as well as their practical applications, is presented in [1]. The paper [2] provides a history of location models over the past 50 years, though it is not exhaustive, as it

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highlights the contributions of only some of the most active European operational research groups. A significant part of scientific research focuses not only on the analysis of object location but also on the evaluation of the market share they occupy, their attractiveness, and competitors' reactions to the appearance of new objects. In particular, [3] describes methods for finding the best locations for competing objects and significant modifications of the classical gravity model. The paper [4] presents mathematical models of optimal location of service centers and partitioning of the territory into service zones, considering the possibility of receiving services from any of the nearest centers. Overlapping zones are provided for cases when the nearest center is unable to provide the service. Centers can include, for example, social protection services, emergency supply warehouses, etc. The distribution of service consumers among service centers is described as optimal multiplex partitioning problems of continuous sets in various formulations – with predefined centers or the need for their location, with or without capacity constraints for service centers. Several possible optimality criteria for multiplex partitioning of sets are proposed. The applied aspects of these problems and related continuous multiple spheres covering problems are considered in [5].

This paper is devoted to the description and theoretical justification of the method for solving continuous problems of optimal multiplex partitioning of sets with center location and capacity constraints.

The object of study is the process of locating service centers and territorial zoning.

The subject of study is mathematical models and methods for optimal location of centers with the determination of their service zones.

The purpose of the work is to organize rationally a network of service centers by determining such locations and service zones to ensure the guaranteed provision of services to consumers in the shortest possible time by assigning them to several nearest centers.

1 PROBLEM STATEMENT

Lets Ω represents the territory of a region where a network of service enterprises operates; $\rho(x)$ is a function that describes the demand for a service at point x within the set Ω; τ_i are points from Ω, which are called centers, $\tau_i \in \Omega, i = 1,2,\ldots,N$; b_i is the capacity of the i-th center $i = 1,2,\ldots,N$; $c(x,\tau_i)$ is the cost of providing the service to a client at the point $x \in \Omega$ by center τ_i, which is considered proportional to the distance between the two points; a_i is the cost of establishing a new center or upgrading an existing one at point τ_i or its fixed organizational costs, calculated per unit of demand, for $i = 1,2,\ldots,N$.

We will consider the problem of optimal location of a certain number of service centers in a given area and assigning service zones to them in such a way that each service consumer (the residents of the area) is assigned to k nearest centers to ensure guaranteed service. The quality criterion is the minimization of transportation (time) and organizational costs, with the condition being the capacity constraints of the centers (the maximum volume of services that a center can provide).

2 REVIEW OF THE LITERATURE

Models of the location of one or several objects among a given set of demand points to achieve a particular goal are considered in [6]. In discrete models, a finite set of potential locations for the objects is given. In continuous models, objects can be located anywhere on the plane or within a region with an infinite number of potential locations. In paper [7], the use of professional optimization software (CPLEX and AIMMS packages) to solve the location-allocation problem is demonstrated, but only discrete models of problems are considered. In article [8], the problems of intermediate transportation hubs, transfer points, and collection warehouses are discussed, highlighting the difference between node location problems and classical object location problems, presenting the p-median problem with single and multiple allocations. Research [9] is aimed at solving continuous location problems for the p-center by repeated analysis. An algorithm based on model relaxation is presented here, enhanced by the addition of four mathematical improvements, which provide a significant reduction in computation time for large data sets (up to one and a half thousand nodes). The flexibility of the improved algorithm is demonstrated, as it can be easily adapted for the α-neighbor p-center problem and problems with constraints.

The features of most of the listed problems are the discrete demand for services, and the obtaining of service zones in the form of spatial monopolies. The classic p-median problem assumes that services are always provided to clients by the nearest facility, whereas in practice, clients often interact with several facilities for various reasons (not just the nearest one). In works [4, 10], distribution rules for modeling such flows are introduced. In [4], a mathematical model of the problem of locating service centers while simultaneously determining overlapping service zones is presented as an optimal multiplex partitioning problem. In [10], the so-called "distributed" p-median problem is formulated, and various types of allocation rules are investigated. For instance, if the weighting coefficients increase (i.e., the assigned flows are larger for objects located farther away), the problem can be solved in polynomial time as a 1-median. For decreasing weights, a special case of which is the classical p-median, a generalization of standard continuous and discrete models is obtained, leading to a broader interpretation of median points. In work [11], the demand for service is continuously distributed over a certain area, as in the problems from [4, 12]. In these works, analytical or spatially interpolated functions are used for the approximate representation of demand, although the results of both methods are subject to significant errors and are characterized by uncertainty. For this reason, [11] introduces a general location model and a continuous Weber

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problem, in which objects can be located anywhere in space to best meet the continuously distributed demand. Due to the complexity of assessing constant demand, it is proposed to integrate optimization methods with the functional capabilities of a geographic information system.

In [12, 13], an approach to developing methods for optimal multiplex partitioning of sets is presented, based on formulating problems in terms of characteristic functions, applying LP-relaxation for the obtained infinite-dimensional optimization problems with Boolean variables, and further using elements of duality theory. Although the relaxation procedure for discrete location-allocation problems is widely used (see, for example, [6, 9]), its application for multiplex partitioning problems in a continuous setting requires theoretical justification and has not been covered in the scientific literature so far. The goal of this work is to rigorously prove statements and theorems that underpin the methods and algorithms for solving continuous optimal multiplex partitioning problems.

3 MATERIALS AND METHODS

To formulate the mathematical model, we introduce some notations and concepts.

$\mathbf{N}$ is a set of all center indices, $\mathbf{N}=\{1,2,\ldots,N\}$; $\mathrm{M}(\mathbf{N},k)$ is a set of all k-element subsets of the set $\mathbf{N}$; L is a number of all k-element subsets of the set $\mathbf{N}$, $L=C_N^k$; Ω_{σ_l} are areas covering clients who have the same k nearest neighboring service centers $\{\tau_{j_1^l},\tau_{j_2^l},\ldots,\tau_{j_k^l}\}$ from N existing (possible), $l=\overline{1,L}$; σ_l is a set of indices $\{j_1^l,j_2^l,\ldots,j_k^l\}$ of centers associated with the subset Ω_{σ_l}.

Definition 1. A collection of subsets $\{\Omega_{\sigma_1},\Omega_{\sigma_2},\ldots,\Omega_{\sigma_L}\}$ from $\Omega\subset E^2$ is called a k-th order partition of the set $\mathbf{\Omega}$ into its subsets $\Omega_{\sigma_1},\ldots,\Omega_{\sigma_L}$, if

$$\bigcup_{l=1}^{L}\Omega_{\sigma_l}=\Omega,\mathrm{mes}\left(\Omega_{\sigma_i}\cap\Omega_{\sigma_j}\right)=0;$$

$$\sigma_i,\sigma_j\in\mathrm{M}(\mathbf{N},k), i\neq j, i,j=\overline{1,L},$$

where $\mathrm{mes}(\cdot)$ – is the measure of the set. Ω_{σ_j} are k-th order subsets of the set $\mathbf{\Omega}$.

Let $\Sigma_\Omega^{N,k}$ – be the class of all possible k-th order partitions of the set $\mathbf{\Omega}$ into its subsets $\Omega_{\sigma_1},\ldots,\Omega_{\sigma_L}$:

$$\Sigma_\Omega^{N,k}=\Big\{\overline{\omega}=\{\Omega_{\sigma_1},\ldots,\Omega_{\sigma_L}\}:\bigcup_{l=1}^{L}\Omega_{\sigma_l}=\Omega,$$
$$mes\left(\Omega_{\sigma_i}\cap\Omega_{\sigma_j}\right)=0,\ \sigma_l,\sigma_j\in\mathrm{M}(\mathbf{N},k),\ i\neq j,\ i,j=\overline{1,L}\Big\}.$$

Problem A. The problem of optimal k-th order partition of the continuous set $\mathbf{\Omega}$ under constraints with location of centers:

$$F\left(\overline{\omega},\boldsymbol{\tau}^N\right)\to\min_{\overline{\omega}\in\Sigma_\Omega^{N,k};\boldsymbol{\tau}^N\in\hat{\Omega}^N}$$

$$F\left(\overline{\omega},\boldsymbol{\tau}^N\right)=\frac{1}{k}\sum_{l=1}^{L}\int_{\Omega_{\sigma_l}}\sum_{i\in\sigma_l}\left(c(x,\tau_i)/w_i+a_i\right)\rho(x)dx,$$

$$\sum_{\substack{l=1\\ l:i\in\sigma_l}}^{L}\int_{\Omega_{\sigma_l}}\gamma_i^l\rho(x)dx=b_i,\quad i=\overline{1,p};\tag{1}$$

$$\sum_{\substack{l=1\\ l:i\in\sigma_l}}^{L}\int_{\Omega_{\sigma_l}}\gamma_i^l\rho(x)dx\le b_i,\quad i=\overline{p+1,N}.\tag{2}$$

Here $x=\left(x^{(1)},x^{(2)}\right)\in\Omega$; $c(x,\tau_i)$, $i=\overline{1,N}$ are bounded functions defined on $\Omega\times\Omega$. The function $\rho(x)$ is bounded and non-negative on $\mathbf{\Omega}$; $w_i>0, a_i\ge0, b_i\ge0, i=\overline{1,N}$, are given constants. The coefficients γ_i^l define the share of the service market that the center τ_i occupies in the territory Ω_{σ_l}, among the facilities $\{\tau_{j_1^l},\tau_{j_2^l},\ldots,\tau_{j_k^l}\}$, serving this territory, such that for all $j=\overline{1,N}$, $l=\overline{1,L}$

$$0\le\gamma_j^l\le1,\ \gamma_{j_1^l}^l+\gamma_{j_2^l}^l+\ldots+\gamma_{j_k^l}^l=1.\tag{3}$$

If we assume that the allocation of demand for services across the entire region $\mathbf{\Omega}$ is proportional to the capacities of the centers, then for all $l=\overline{1,L}$ and for all $j=\overline{1,N}$, $j\in\sigma_l$, the values γ_j^l are given by the following expression: $\gamma_j^l=b_j/\sum_{q:q\in\sigma_l}b_q$. If the demand is distributed evenly among the centers, then $\gamma_j^l=\frac{1}{k}$, for all j and l.

Lemma 1 (see [13]). Let $S=\int_\Omega\rho(x)dx$ in problem **A**, and assume the conditions hold:

$$0\le b_i\le S,\quad i=1,\ldots,N;\ \sum_{i=1}^{p}b_i\le S\le\sum_{i=1}^{N}b_i.\tag{4}$$

Then, for any fixed vector $\tau^N\in\Omega^N$ the set of feasible partitions (satisfying conditions (1), (2)) is non-empty.

The method for solving Problem **A** involves expressing it in terms of characteristic vector-functions of the k-th order partition of the set $\mathbf{\Omega}$.

Definition 2. A characteristic vector-function of the k-th order partition $\overline{\omega}=\{\Omega_{\sigma_1},\ldots,\Omega_{\sigma_l},\ldots,\Omega_{\sigma_L}\}$ of the set $\mathbf{\Omega}$ is the vector-function $\chi(\cdot)=\left(\chi_1(\cdot),\ldots,\chi_l(\cdot),\ldots,\chi_L(\cdot)\right)$, defined on $\mathbf{\Omega}$, which components are characteristic

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functions of the subsets Ω_{σ_l} and are given by the formula (almost everywhere) for $x \in \Omega$

$$\chi_l(x)=\begin{cases}1, \; x\in\Omega_{\sigma_l},\\ 0, \; x\in\Omega\setminus\Omega_{\sigma_l}, \quad l=\overline{1,L}.\end{cases}$$

In [12, 13], to describe the *k*-th order partition of the set **Ω**, and *NL*-dimensional vector $\lambda(x)$ is introduced with coordinates

$$\lambda_i^l(x)=\begin{cases}1, \text{ if } x\in\Omega_{\sigma_l} \text{ and } i\in\sigma_l,\\ 0 \text{ otherwise}, \; i=\overline{1,N}, i=\overline{1,L},\end{cases} \tag{5}$$

where $\sigma_l=\left\{j_1^l,...,j_k^l\right\}$ is the set of indices of the centers $\left\{\tau_{j_1^l},...,\tau_{j_k^l}\right\}$, associated with Ω_{σ_l}.

Using (5), the coordinates of the function $\chi(\cdot)$ can be represented as $\chi_l(x)=\prod_{i\in\sigma_l}\lambda_i^l(x)$, $l=\overline{1,L}$. for each point $x\in\Omega$. Since each point $x\in\Omega$ belongs to only one of the subsets Ω_{σ_l}, among all components $\lambda_i^l(x)$ only *k* components for the same index *l* are equal to one. This means that the vector-functions $\lambda(\cdot)$ and $\chi(\cdot)$, which define the same *k*-th order partition $\bar{\omega}$ of the set **Ω**, satisfy the following conditions: for each point $x \in \Omega$

$$\chi_l(x)=0\vee 1, \; \lambda_i^l(x)=0\vee 1, \; i=\overline{1,N}, \; l=\overline{1,L},$$

$$\sum_{i=1}^{N}\lambda_i^l(x)=k\chi_l(x), \; l=\overline{1,L}; \sum_{l=1}^{L}\chi_l(x)=1.$$

The relationship between $\lambda(\cdot)$ and $\chi(\cdot)$ is one-to-one. Indeed, if $x\in\Omega_{\sigma_l}$, i.e., $\chi_l(x)=1$, then in the corresponding vector $\lambda(x)$ only *k* components will be equal to one:

$$\lambda_i^l=\begin{cases}1, \; i\in\sigma_l,\\ 0, i\in\mathrm{N}\setminus\sigma_l,\end{cases} \quad i=\overline{1,N}. \tag{6}$$

The remaining components $\lambda_i^t(x)=0, \forall i=\overline{1,N}$, $t=\overline{1,L}, t\neq l$.

On the other hand, if a point $x\in\Omega$ is associated with a vector $\lambda(x)=\left(\lambda^1(x),...,\lambda^l(x),...,\lambda^L(x)\right)$, in which only *k* components are equal to one among $\lambda_1^l(x),...,\lambda_N^l(x)$, and the rest are zero, this means that there exists a set $\sigma_l=\left\{j_1^l,j_2^l,...,j_k^l\right\}$, such that $x\in\Omega_{\sigma_l}$ and $\chi_l(x)=1$. From a practical perspective, $\lambda_i^l(x)$, $i=\overline{1,N}$, serves as an indicator of whether a client at point *x* is served by center τ_i along with the remaining (*k*–1)-centers τ_j, where $j\in\sigma_l$.

To formulate the OMPS problem in terms of the characteristic functions of the partition, both vectors will be used, with the vector-function $\chi(\cdot)$ considered as the unknown (which distinguishes this work from [12, 13]). In the vector $\lambda^l(x)=\left(\lambda_1^l(x),...,\lambda_N^l(x)\right)$, which corresponds to the component $\chi_l(x)$ of the characteristic vector-function of the partition, the argument will be omitted for compactness of notation. This vector defines the indicators of the indices in the set σ_l from the **N** (see formula (6)) and will therefore be used as a template.

Problem **A** is formulated in the following equivalent form.

Problem B.

$$\min_{(\chi(\cdot),\boldsymbol{\tau}^N)\in\Gamma^k\times\hat{\Omega}^N} I\left(\chi(\cdot),\boldsymbol{\tau}^N\right),$$

where

$$I\left(\chi(\cdot),\boldsymbol{\tau}^N\right)=\frac{1}{k}\int_{\Omega}\sum_{l=1}^{L}\left(\sum_{i=1}^{N}\left(c(x,\tau_i)/w_i+a_i\right)\lambda_i^l\right)\rho(x)\chi_l(x)dx$$

$$\Gamma^k=\Big\{\chi(\cdot):\chi(\cdot)\in\Gamma_0^k,$$

$$\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\rho(x)\chi_l(x)dx=b_i, \quad i=1,\dots,p,$$

$$\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\rho(x)\chi_l(x)dx\le b_i, \quad i=p+1,\dots,N\Big\};$$

$$\Gamma_0^k=\Big\{\chi(\cdot)=\left(\chi_1(\cdot),\dots,\chi_L(\cdot)\right):\chi_l(x)=0\vee 1,$$

$$l=\overline{1,L},\sum_{l=1}^{L}\chi_l(x)=1 \text{ a.e. } x\in\Omega\Big\}.$$

To solve Problem **B** with Boolean variables, we perform its LP relaxation.

Problem C:

$$\min_{(\chi(\cdot),\boldsymbol{\tau}^N)\in\Gamma_2^k\times\hat{\Omega}^N} I\left(\chi(\cdot),\boldsymbol{\tau}^N\right),$$

where

$$\Gamma_2^k=\Big\{\chi(\cdot):\chi(\cdot)\in\Gamma_1^k,$$

$$\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\chi_l(x)\rho(x)dx=b_i, \quad i=\overline{1,p},$$

$$\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\chi_l(x)\rho(x)dx\le b_i, \quad i=\overline{p+1,N}\Big\};$$

$$\Gamma_1^k=\Big\{\chi(\cdot)=\left(\chi_1(\cdot),\dots,\chi_l(\cdot),\dots,\chi_L(\cdot)\right):0\le\chi_l(x)\le 1,$$

$$l=\overline{1,L};\; \sum_{l=1}^{L}\chi_l(x)=1 \text{ a.e. for } x\in\Omega\Big\}.$$

Justification of the reduction of Problem **B** to Problem **C.** From the fact that $\Gamma_0^k\subset\Gamma_1^k$, it follows that $\Gamma^k\subset\Gamma_2^k$.

Statement 1. Γ_2^k is a bounded, closed, and convex set in the Hilbert space $L_2^L(\Omega)$ with the norm $\|\chi(\cdot)\|=\sqrt{\int_{\Omega}\sum_{l=1}^{L}\left[\chi_l(x)\right]^2dx}$.

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Proof. Let $\hat{\chi}(\cdot)$ and $\overline{\chi}(\cdot)$ be arbitrary elements of the set Γ_2^k, and let α be a constant such that $0 \le \alpha \le 1$. We will show that $\alpha\hat{\chi}_l(x)+(1-\alpha)\overline{\chi}_l(x) \in \Gamma_2^k$. Indeed, almost everywhere for $x \in \Omega$

$$\sum_{l=1}^{L}\left(\alpha\hat{\chi}_l(x)+(1-\alpha)\overline{\chi}_l(x)\right)=$$

$$=\alpha\sum_{l=1}^{L}\hat{\chi}_l(x)+(1-\alpha)\sum_{l=1}^{L}\overline{\chi}_l(x)=\alpha+(1-\alpha)=1.$$

For each $i=1,\ldots,p$:

$$\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\left(\alpha\hat{\chi}_l(x)+(1-\alpha)\overline{\chi}_l(x)\right)\rho(x)dx=$$

$$=\alpha\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\hat{\chi}_l(x)\rho(x)dx+(1-\alpha)\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\overline{\chi}_l(x)\rho(x)dx=$$

$$=\alpha b_i+(1-\alpha)b_i=b_i.$$

For each $i=p+1,\ldots,N$:

$$\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\left(\alpha\hat{\chi}_l(x)+(1-\alpha)\overline{\chi}_l(x)\right)\rho(x)dx=$$

$$=\alpha\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\hat{\chi}_l(x)\rho(x)dx+(1-\alpha)\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\overline{\chi}_l(x)\rho(x)dx\le$$

$$\le\alpha b_i+(1-\alpha)b_i=b_i.$$

Since for $x \in \Omega$ $0 \le \hat{\chi}_l(x),\overline{\chi}_l(x) \le 1$, it follows that $0 \le \alpha\hat{\chi}_l(x)+(1-\alpha)\overline{\chi}_l(x) \le 1$, $l=\overline{1,L}$. Thus, the convexity of the set Γ_2^k is proven.

Let the sequence $\{\chi^{(m)}(\cdot)\} \in \Gamma_2^k$ converge to some function $\chi(\cdot)$ in the norm of the space $L_2^L(\Omega)$. Consider a subsequence $\{\chi^{(m_r)}(\cdot)\}$, that converges to $\chi(\cdot)$ almost everywhere on Ω. For $\chi^{(m_r)}(\cdot) \in \Gamma_2^k$ the following conditions hold:

$$\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\rho(x)\chi_l^{(m_r)}(x)dx=b_i,\quad i=1,2,\ldots,p;$$

$$\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\rho(x)\chi_l^{(m_r)}(x)dx\le b_i,\quad i=p+1,\ldots,N;$$

$$0\le\chi_l^{(m_r)}(x)\le 1, l=\overline{1,L};$$

$$\sum_{l=1}^{L}\chi_l^{(m_r)}(x)=1,\ \text{a.e. for } x\in\Omega.$$

Taking into account that the function $\rho(x)$ is bounded, measurable, and non-negative on the set **Ω**, according to the Dominated Convergence Theorem [14], the limit transition holds as $m_r \to \infty$

$$\lim_{m_r\to\infty}\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\rho(x)\chi_l^{(m_r)}(x)dx=$$

$$=\int_{\Omega}\lim_{m_r\to\infty}\left(\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\rho(x)\chi_l^{(m_r)}(x)\right)dx=$$

$$=\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\rho(x)\left(\lim_{m_r\to\infty}\chi_l^{(m_r)}(x)\right)dx=\int_{\Omega}\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\rho(x)\chi_l^{(m_r)}(x)dx.$$

Thus, we conclude that all the above conditions are satisfied with the function $\chi(\cdot)$. This means that the set Γ_2^k is closed.

Let $\chi(\cdot)\in\Gamma_2^k$, then $0\le\chi_l(x)\le 1$ a. e. for $x\in\Omega, l=\overline{1,L}$,

$$\|\chi(\cdot)\|=\sqrt{\int_{\Omega}\sum_{l=1}^{L}[\chi_l(x)]^2dx}\le\sqrt{\int_{\Omega}\sum_{l=1}^{L}1^2dx}=\sqrt{Lmes(\Omega)}=Const.$$

The boundedness of Γ_2^k is proven.

Lemma 2. For each fixed vector $\boldsymbol{\tau}^N\in\hat{\Omega}^N$, the functional $\overline{I}(\chi(\cdot))=I(\chi(\cdot),\boldsymbol{\tau}^N)$ in Problem **A** is linear and continuous on the set Γ_2^k.

Proof. Let $\boldsymbol{\tau}^N$ be an arbitrary but fixed vector from $\hat{\Omega}^N$. Define the quantity

$$q=\sup_{x\in\Omega}\max_{i=1,N}\left(c(x,\tau_i)/w_i+a_i\right)\rho(x).$$

The linearity of the functional $\overline{I}(\chi(\cdot))$ is obvious because of the linearity of the Lebesgue integral in the functional $I(\chi(\cdot),\boldsymbol{\tau}^N)$.

To prove the boundedness of $\overline{I}(\chi(\cdot))$, we use Hölder's inequality:

$$\left|\overline{I}(\chi(\cdot))\right|=\left|I(\chi(\cdot),\boldsymbol{\tau}^N)\right|=$$

$$=\left|\frac{1}{k}\int_{\Omega}\sum_{l=1}^{L}\sum_{i=1}^{N}\left(c(x,\tau_i)/w_i+a_i\right)\lambda_i^l\chi_l(x)\rho(x)dx\right|\le$$

$$\le\left|\frac{1}{k}\cdot q\int_{\Omega}\sum_{l=1}^{L}\sum_{i=1}^{N}\lambda_i^l\chi_l(x)dx\right|=\left|q\int_{\Omega}\sum_{l=1}^{L}\chi_l(x)dx\right|\le$$

$$\le q\sqrt{\int_{\Omega}\sum_{l=1}^{L}[\chi_l(x)]^2dx}=q\cdot\|\chi(\cdot)\|_{L_2^L(\Omega)}.$$

Thus, the functional $\overline{I}(\chi(\cdot))$ is linear, bounded, and, according to [14], continuous with respect to $\chi(\cdot)$.

Statement 2. If the conditions (4) hold for Problem **A**, then for each fixed vector $\boldsymbol{\tau}^N\in\hat{\Omega}^N$ Problem **C** is solvable with respect to $\chi(\cdot)$.

Proof. Let $\boldsymbol{\tau}^N$ be an arbitrary but fixed vector from $\hat{\Omega}^N$ in Problem **A** and let the conditions (4) hold. According to Lemma 1, the set of feasible partitions is non-empty, meaning the set Γ^k of admissible vector-functions

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$\chi(\cdot)$ in Problem **C** is also no-empty. According to Lemma 2, the functional $I(\chi(\cdot),\boldsymbol{\tau}^N)$ is linear (convex) and continuous with respect to $\chi(\cdot)$ on the set Γ_2^k. According to Statement 1, Γ_2^k is convex, closed, and bounded, and therefore, by the generalized Weierstrass theorem [14], the functional $I\left(\chi(\cdot),\boldsymbol{\tau}^N\right)$ attains its exact lower bound on Γ_2^k. Thus, Problem **C** is solvable with respect to $\chi(\cdot)$ for each fixed $\boldsymbol{\tau}^N \in \hat{\Omega}^N$.

Statement 3. Let $\overline{\boldsymbol{\tau}}^N \in \hat{\Omega}^N$ be an arbitrary but fixed vector. Among the set of points in Γ_2^k, where the functional $\overline{I}(\chi(\cdot)) = I\left(\chi(\cdot),\overline{\boldsymbol{\tau}}^N\right)$ attains its minimum, there exists at least one extreme point of Γ_2^k.

Proof. By the generalized Weierstrass theorem, the continuous (see Lemma 2) functional $\overline{I}(\chi(\cdot))$ on the convex, closed, and bounded set Γ_2^k (see Statement 1) attains its exact lower bound, and the set Γ^* of its minimum points is non-empty, convex, and bounded, making it weakly compact in $L_2^L(\Omega)$. According to the Krein-Milman theorem [15], such a set Γ^* has an extreme point, denoted as $\chi^*(\cdot)$. We now show that $\chi^*(\cdot)$ is an extreme point of the set Γ_2^k.

Assume the contrary. Then it can be expressed as a linear combination of two points $u(\cdot),v(\cdot) \in \Gamma_2^k \setminus \Gamma^*, u \neq v$: $\chi^*(\cdot) = \alpha u(\cdot) + (1-\alpha)v(\cdot)$, where $0<\alpha<1$. Since $\chi^*(\cdot)$ is a minimum point of the functional, it follows that $\overline{I}(u(\cdot)) \geq \overline{I}(\chi^*(\cdot)),\ \overline{I}(v(\cdot)) \geq \overline{I}(\chi^*(\cdot))$. Due to its linearity

$$\overline{I}(\chi^*(\cdot)) = \alpha\overline{I}(u(\cdot)) + (1-\alpha)\overline{I}(v(\cdot)) =$$
$$= \overline{I}(v(\cdot)) + \alpha(\overline{I}(u(\cdot)) - \overline{I}(v(\cdot))).$$

This is possible only if $\overline{I}(\chi^*(\cdot)) = \overline{I}(u(\cdot)) = \overline{I}(v(\cdot))$. Thus $u(\cdot),v(\cdot) \in \Gamma^*$. This leading to a contradiction. We conclude that $\chi^*(\cdot)$ is an extreme point of Γ^*.

Statement 3 is proven.

Statement 4. Any extreme point of the set Γ_2^k is a characteristic vector-function of some k-th order partition $\overline{\omega} = \left\{\Omega_{\sigma_1},...,\Omega_{\sigma_l},\dots,\Omega_{\sigma_L}\right\}$ of the set Ω.

Proof. Assume the contrary: let $\chi(\cdot)$ be an extreme point of the set Γ_2^k, but at least one of its components $\chi_s(x), 1 \leq s \leq L$, is not a characteristic function of the corresponding subset Ω_{σ_s} in the *k*-th order partition of the set **Ω**. Without loss of generality, assume that this is the case for the function $\chi_1(x)$. This means that there exists a subset $\mathrm{P} \subset \Omega$, $mes(\mathrm{P}) > 0$, such that for all $x \in \mathrm{P}$: $\delta < \chi_1(x) < 1-\delta,\ 0<\delta<1$.

Introduce an auxiliary function $\mu(\cdot)$, that satisfies the following conditions:

1) for all $x \in \Omega \setminus \mathrm{P}$ $\mu(x) = 0$;
2) for all $x \in \mathrm{P}$

$$\left|\mu(x)\right| < \delta \tag{7}$$

3) where $x \in \mathrm{P}$ $\mu(\cdot)$ is a nontrivial solution to the system of equations.

$$\int_{\mathrm{P}} \gamma_i^1 \lambda_i^1 \rho(x)\mu(x)dx = 0,\ i = 1,2,\dots,N \tag{8}$$

$$\int_{\mathrm{P}} \sum_{l=2}^{L} \gamma_i^l \lambda_i^l \rho(x) \frac{\chi_l(x)}{\sum\limits_{j=2}^{L} \chi_j(x)} \mu(x)dx = 0,\ i = 1,2,\dots,N. \tag{9}$$

The function $\mu(\cdot)$, which satisfies conditions (1)–(3), can be constructed, for example, using the following approach. To partition P into $(2N+1)$ non-overlapping subsets of nonzero measure and define $\mu(\cdot)$ as a piecewise constant function on P, taking a constant value on each subset. The corresponding values can be found as a nontrivial solution to system (8), (9) – a system of $2N$ linear homogeneous algebraic equations with respect to $(2N+1)$ variables. The obtained values are then normalized to satisfy condition (7).

Using the function $\mu(\cdot)$, we construct two vector-functions $\hat{\chi}(\cdot)$ and $\overline{\chi}(\cdot)$ as follows: for all $x \in \Omega$

$$\hat{\chi}_1(x) = \chi_1(x) + \mu(x),$$
$$\hat{\chi}_j(x) = \chi_j(x) - \frac{\chi_j(x)}{\sum\limits_{l=2}^{L} \chi_l(x)} \mu(x),\ j = \overline{2,L};$$
$$\overline{\chi}_1(x) = \chi_1(x) - \mu(x),$$
$$\overline{\chi}_j(x) = \chi_j(x) + \frac{\chi_j(x)}{\sum\limits_{l=2}^{L} \chi_l(x)} \mu(x),\ j = \overline{2,L};$$

We will show that $\hat{\chi}(\cdot)$ and $\overline{\chi}(\cdot) \in \Gamma_2^k$. Indeed, due to conditions (8) and (9), for all $i = 1,2,\dots,N$:

$$\int_{\Omega} \sum_{l=1}^{L} \gamma_i^l \lambda_i^l \rho(x)\hat{\chi}_l(x)dx = \int_{\Omega\setminus\mathrm{P}} \sum_{l=1}^{L} \gamma_i^l \lambda_i^l \rho(x)\chi_l(x)dx +$$
$$\int_{\mathrm{P}} (\gamma_i^1 \lambda_i^1 \left(\chi_1(x) + \mu(x)\right) dx +$$
$$+\int_{\mathrm{P}} \sum_{l=2}^{L} \gamma_i^l \lambda_i^l (\chi_l(x) - \frac{\chi_l(x)}{\sum_{j=2}^{L} \chi_j(x)} \mu(x)))\rho(x)dx =$$

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$$= \int_{\Omega \setminus P} \sum_{l=1}^{L} \gamma_i^l \lambda_i^l \rho(x) \chi_l(x) dx +$$

$$+ \int_P \left(\gamma_i^1 \lambda_i^1 \chi_1(x) + \sum_{l=2}^{L} \gamma_i^l \lambda_i^l \chi_l(x) \right) \rho(x) dx +$$

$$+ \int_P \gamma_i^1 \lambda_i^1 \mu(x) \rho(x) dx - \int_P \sum_{l=2}^{L} \gamma_i^l \lambda_i^l \frac{\chi_l(x)}{\sum_{j=2}^{L} \chi_j(x)} \rho(x) \mu(x) dx =$$

$$= \int_{\Omega} \sum_{l=1}^{L} \gamma_i^l \lambda_i^l \rho(x) \chi_l(x) dx .$$

Similarly,

$$\int_{\Omega} \sum_{l=1}^{L} \gamma_i^l \lambda_i^l \rho(x) \overline{\chi}_l(x) dx = \int_{\Omega} \sum_{l=1}^{L} \gamma_i^l \lambda_i^l \rho(x) \chi_l(x) dx, i = \overline{1, N} .$$

By direct verification, we confirm the validity of the equalities $\sum_{l=1}^{L} \hat{\chi}_l(x) = 1, \sum_{l=1}^{L} \overline{\chi}_l(x) = 1$. For example, for $\overline{\chi}(\cdot)$ we have:

$$\sum_{l=1}^{L} \overline{\chi}_l(x) = \chi_1(x) - \mu(x) + \sum_{l=2}^{L} (\chi_l(x) + \frac{\chi_j(x)}{\sum_{j=2}^{L} \chi_j(x)} \mu(x)) =$$

$$= \sum_{l=1}^{L} \chi_l(x) + \frac{1}{\sum_{j=2}^{L} \chi_j(x)} (-\mu(x) \sum_{j=2}^{L} \chi_j(x) + \mu(x) \sum_{l=2}^{L} \chi_l(x)) =$$

$$= \sum_{l=1}^{L} \chi_l(x) = 1.$$

By the definition of the functions $\hat{\chi}_1(\cdot)$, $\overline{\chi}_1(\cdot)$ and condition (7), it follows that: $\hat{\chi}_1(\cdot) \geq 0$, $\overline{\chi}_1(\cdot) \geq 0$ for all $x \in \Omega$. The remaining components of the vector-functions $\hat{\chi}_1(\cdot), \overline{\chi}_1(\cdot)$ are also non-negative. Indeed, considering that $\chi(\cdot) \in \Gamma_2^k$, for $j = \overline{2, L}$ we obtain:

$$\hat{\chi}_j(x) = \chi_j(x) - \frac{\chi_j(x)}{\sum_{l=2}^{L} \chi_l(x)} \mu(x) \geq$$

$$\geq \chi_j(x) - \frac{\chi_j(x)}{\delta} |\mu(x)| \geq \chi_j(x) - \frac{\chi_j}{\delta} \cdot \delta = 0.$$

Similarly, we verify the inequality $\overline{\chi}_j(x) \geq 0, j = \overline{2, L}$.

By construction and the previously derived relationships, for all $x \in \Omega$: $0 \leq \hat{\chi}_i(x) \leq 1$, $0 \leq \overline{\chi}_i(x) \leq 1, \forall l = \overline{1, L}$.

Thus, it is proven that $\hat{\chi}(\cdot), \overline{\chi}(\cdot) \in \Gamma_2^k$. However, the representation $\chi(\cdot) = \frac{1}{2} \hat{\chi}(\cdot) + \frac{1}{2} \overline{\chi}(\cdot)$ contradicts the statement that $\chi(\cdot)$ is an extreme point of the set Γ_2^k. Therefore $\chi(\cdot)$ must be a characteristic vector-function of some k-th order partition of the set $\boldsymbol{\Omega}$. Statement 4 is proven.

Thus, according to Statement 3, among the points of the set Γ_2^k, where the linear functional $I(\chi(\cdot), \boldsymbol{\tau}^N)$ attains its minimum value for a fixed vector $\boldsymbol{\tau}^N \in \hat{\Omega}^N$ there exists at least one extreme point of Γ_2^k. According to Statement 2, extreme points are characteristic vector-functions of subsets $\Omega_{\sigma_l}, l = \overline{1, L}$, which form a k-th order partition of the set $\boldsymbol{\Omega}$ for each fixed vector $\boldsymbol{\tau}^N \in \hat{\Omega}^N$. Thus, the set of optimal solutions to Problem **C** includes the optimal solutions to Problem **B**, which means that Problem **B** can be reduced to Problem **C** by selecting from the solutions of the latter those that are also solutions to Problem **B**. This reduction forms the basis of optimal multiplex partitioning methods, and the necessary and sufficient conditions for optimal multiplex partitioning have been obtained.

Optimality conditions for the solution of Problem C. We construct the Lagrange functional for Problem **C**:

$$W((\chi(\cdot), \boldsymbol{\tau}^N), (\psi_0(\cdot), \boldsymbol{\psi})) =$$

$$= \frac{1}{k} \int_{\Omega} \sum_{l=1}^{L} \sum_{i=1}^{M} \left(c(x, \tau_i) / w_i + a_i \right) \lambda_i^l \chi_l(x) \rho(x) dx +$$

$$+ \sum_{i=1}^{N} \psi_i (\int_{\Omega} \sum_{l=1}^{L} \gamma_i^l \lambda_i^l \chi_l(x) \rho(x) dx - b_i) +$$

$$+ \int_{\Omega} \psi_0(x) (\sum_{l=1}^{L} \chi_l(x) - 1) dx =$$

$$= \int_{\Omega} \sum_{l=1}^{L} (\sum_{i=1}^{N} ((\frac{c(x, \tau_i)}{w_i} + a_i) / k + \gamma_i^l \psi_i) \lambda_i^l \rho(x) + \psi_0(x)) \chi_l(x) dx -$$

$$- \int_{\Omega} \psi_0(x) dx - \sum_{i=1}^{N} \psi_i b_i .$$

The functional $W\left(\left(\chi(\cdot), \boldsymbol{\tau}^N\right), \left(\psi_0(\cdot), \boldsymbol{\psi}\right)\right)$ is defined on the Cartesian product $\left(\Lambda \times \hat{\Omega}^N\right) \times \left(\Phi \times \Psi\right)$, where

$$\Lambda = \left\{ \chi(\cdot) \in L_2^L(\Omega) : 0 \leq \chi_l(x) \leq 1 \ \forall x \in \Omega, l = \overline{1, L} \right\};$$

$$\Phi = \left\{ \psi_0 : \psi_0(\cdot) \in L_2(\Omega) \right\};$$

$$\Psi = \left\{ \boldsymbol{\psi} \in R^N : \psi_i \geq 0, i = p+1, \ldots, N \right\}.$$

Definition 3. A pair $\left(\left(\hat{\chi}(\cdot), \hat{\boldsymbol{\tau}}^N\right), \left(\hat{\psi}_0(\cdot), \hat{\boldsymbol{\psi}}\right)\right)$ is called a saddle point of the Lagrange functional $W((\chi(\cdot), \boldsymbol{\tau}^N), (\psi_0(\cdot), \boldsymbol{\psi}))$ on the set $\left(\Lambda \times \hat{\Omega}^N\right) \times \left(\Phi \times \Psi\right)$, if for all $\left(\chi(\cdot), \boldsymbol{\tau}^N\right) \in \Lambda \times \hat{\Omega}^N$ and for all $\left(\psi_0(\cdot), \boldsymbol{\psi}\right) \in \Phi \times \Psi$, the following inequalities hold:

$$W\left(\left(\hat{\chi}(\cdot), \hat{\boldsymbol{\tau}}^N\right), \left(\psi_0(\cdot), \boldsymbol{\psi}\right)\right) \leq W\left(\left(\hat{\chi}(\cdot), \hat{\boldsymbol{\tau}}^N\right), \left(\hat{\psi}_0(\cdot), \hat{\boldsymbol{\psi}}\right)\right);$$

$$W\left(\left(\hat{\chi}(\cdot), \hat{\boldsymbol{\tau}}^N\right), \left(\hat{\psi}_0(\cdot), \hat{\boldsymbol{\psi}}\right)\right) \leq W\left(\left(\chi(\cdot), \boldsymbol{\tau}^N\right), \left(\hat{\psi}_0(\cdot), \hat{\boldsymbol{\psi}}\right)\right).$$

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The problem dual to Problem **C** is formulated as follows:

$$H(\psi_0(\cdot),\boldsymbol{\psi}) \to \max_{(\psi_0(\cdot),\boldsymbol{\psi})\in\Phi\times\Psi}$$
$$H(\psi_0(\cdot),\boldsymbol{\psi}) = \min_{(\chi,\boldsymbol{\tau}^N)\in\Lambda\times\hat{\Omega}^N} W((\chi(\cdot),\boldsymbol{\tau}^N),(\psi_0(\cdot),\boldsymbol{\psi})) \quad (10)$$

Theorem 1 (see [16]). For Problem **C** and (10) to be related by the duality relation $I_* = W^*$, and for the supremum in (10) to be attainable, it is necessary and sufficient that a saddle point of the functional $W((\chi(\cdot),\boldsymbol{\tau}^N),(\psi_0(\cdot),\boldsymbol{\psi}))$ exists on the set $(\Lambda\times\hat{\Omega}^N)\times(\Phi\times\Psi)$.

Thus, solving the pair of problems **C** and (10) is equivalent to finding a saddle function of the functional $W((\chi(\cdot),\boldsymbol{\tau}^N),(\psi_0(\cdot),\boldsymbol{\psi}))$ on the set $(\Lambda\times\hat{\Omega}^N)\times(\Phi\times\Psi)$.

Let the vectors $\overline{\boldsymbol{\psi}}\in\Psi$ and $\overline{\boldsymbol{\tau}}^N\in\hat{\Omega}^N$ be arbitrary but fixed in the functional $H(\psi_0(\cdot),\boldsymbol{\psi})$. Consider the problem

$$W\left(\left(\chi(\cdot),\overline{\boldsymbol{\tau}}^N\right),\left(\psi_0(\cdot),\overline{\boldsymbol{\psi}}\right)\right) \to \min_{\chi(\cdot)\in\Lambda}\max_{\psi_0(\cdot)\in\Phi}. \quad (11)$$

Further, for brevity, we will use the following notation: $d_i(x) = \left(c(x,\tau_i)/w_i + a_i\right)/k$, $i=\overline{1,N}$.

The center τ_i may be fixed or not, depending on the context.

For each arbitrarily fixed point $x\in\Omega$ we introduce a function of $(L+1)$ variables:

$$Q\left(\chi(x),\psi_0(x)\right) =$$
$$= \sum_{l=1}^{L}\left[\sum_{i=1}^{N}\left(d_i(x)+\gamma_i^l\psi_i\right)\lambda_i^l\rho(x)+\psi_0(x)\right]\chi_l(x) - \psi_0(x).$$

This function is defined on the Cartesian product $\Lambda_x\times\Phi_x$ of the projections of the sets Λ and Φ for $x\in\Omega$.

Theorem 2. For an admissible pair $\left(\hat{\chi}(\cdot),\hat{\psi}_0(\cdot)\right)\in\Lambda\times\Phi$ to be a solution of problem (11), it is necessary and sufficient that almost everywhere for $x\in\Omega$, the following condition holds:

$$Q\left(\hat{\chi}(x),\hat{\psi}_0(x)\right) = \max_{\psi_0(x)\in\Phi_x}\min_{\chi(x)\in\Lambda_x} Q\left(\chi(x),\psi_0(x)\right). \quad (12)$$

Proof. Necessity. Let $\left(\hat{\chi}(\cdot),\hat{\psi}_0(\cdot)\right)\in\Lambda\times\Phi$ be an optimal solution to problem (11), i.e., $\forall\chi(\cdot)\in\Lambda,\ \psi_0(\cdot)\in\Phi$

$$W\left(\left(\hat{\chi}(\cdot),\overline{\boldsymbol{\tau}}^N\right),\left(\psi_0(\cdot),\overline{\boldsymbol{\psi}}\right)\right) \le W\left(\left(\hat{\chi}(\cdot),\overline{\boldsymbol{\tau}}^N\right),\left(\hat{\psi}_0(\cdot),\overline{\boldsymbol{\psi}}\right)\right) \quad (13)$$
$$W\left(\left(\hat{\chi}(\cdot),\overline{\boldsymbol{\tau}}^N\right),\left(\hat{\psi}_0(\cdot),\overline{\boldsymbol{\psi}}\right)\right) \le W\left(\left(\chi(\cdot),\overline{\boldsymbol{\tau}}^N\right),\left(\hat{\psi}_0(\cdot),\overline{\boldsymbol{\psi}}\right)\right) \quad (14)$$

We will show that almost everywhere for $x\in\Omega$ this pair $\left(\hat{\chi}(\cdot),\hat{\psi}_0(\cdot)\right)$ satisfies condition (12). Assume the contrary: there exists a subset $\tilde{\Omega}$ of the set Ω such that $mes(\tilde{\Omega})>0$ and $\forall x\in\tilde{\Omega}$ condition (12) does not hold, i.e., $\forall x\in\tilde{\Omega}$ there exist $\tilde{\chi}(x)\in\Lambda_x$, for which the following inequality is valid:

$$Q\left(\tilde{\chi}(x),\hat{\psi}_0(x)\right) < Q\left(\hat{\chi}(x),\hat{\psi}_0(x)\right).$$

We construct a new pair of admissible functions for the problem (11):

$$\left(\overline{\chi}(x),\hat{\psi}_0(x)\right) = \begin{cases}\left(\tilde{\chi}(x),\hat{\psi}_0(x)\right)\in\Lambda_x\times\Phi_x, & \forall x\in\tilde{\Omega},\\ \left(\hat{\chi}(x),\hat{\psi}_0(x)\right)\in\Lambda_x\times\Phi_x, & \forall x\in\Omega\setminus\tilde{\Omega}.\end{cases}$$

Integrating $Q\left(\overline{\chi}(x),\hat{\psi}_0(x)\right)$ over the entire region Ω and adding the constant $\left(-\sum_{i=1}^{N}\overline{\psi}_i b_i\right)$, we obtain:

$$W\left(\left(\overline{\chi}(\cdot),\overline{\boldsymbol{\tau}}^N\right),\left(\hat{\psi}_0(\cdot),\overline{\boldsymbol{\psi}}\right)\right) = \int_\Omega Q\left(\overline{\chi}(x),\hat{\psi}_0(x)\right)dx - \sum_{i=1}^{N}\overline{\psi}_i b_i =$$
$$= \int_{\tilde{\Omega}} Q\left(\tilde{\chi}(x),\hat{\psi}_0(x)\right)dx + \int_{\Omega\setminus\tilde{\Omega}} Q\left(\hat{\chi}(x),\hat{\psi}_0(x)\right)dx - \sum_{i=1}^{N}\overline{\psi}_i b_i.$$

Similarly splitting the integral in $W\left(\left(\chi(\cdot),\overline{\boldsymbol{\tau}}^N\right),\left(\psi_0(\cdot),\overline{\boldsymbol{\psi}}\right)\right)$ with $\chi(\cdot)=\hat{\chi}(\cdot),\psi_0(\cdot)=\hat{\psi}_0(\cdot)$ and comparing the right-hand sides of the obtained relations, we conclude:

$$W\left(\left(\overline{\chi}(\cdot),\overline{\boldsymbol{\tau}}^N\right),\left(\hat{\psi}_0(\cdot),\overline{\boldsymbol{\psi}}\right)\right) < W\left(\left(\hat{\chi}(\cdot),\overline{\boldsymbol{\tau}}^N\right),\left(\hat{\psi}_0(\cdot),\overline{\boldsymbol{\psi}}\right)\right),$$

which contradicts (14).

We can also assume the existence of a subset $\breve{\Omega}$ of the set Ω such that $mes(\breve{\Omega})>0$ and $\forall x\in\breve{\Omega}$, there exists $\breve{\psi}_0(x)\in\Phi_x$ such that the following inequality holds:

$$Q\left(\hat{\chi}(x),\hat{\psi}_0(x)\right) < Q\left(\hat{\chi}(x),\breve{\psi}_0(x)\right).$$

Then, for the pair $\left(\hat{\chi}(x),\overline{\overline{\psi}}_0(x)\right)$, defined as follows

$$\left(\hat{\chi}(x),\overline{\overline{\psi}}_0(x)\right) = \begin{cases}\left(\hat{\chi}(x),\breve{\psi}_0(x)\right)\in\Lambda_x\times\Phi_x, & \forall x\in\breve{\Omega}\\ \left(\hat{\chi}(\cdot),\hat{\psi}_0(\cdot)\right)\in\Lambda_x\times\Phi_x, & \forall x\in\Omega\setminus\breve{\Omega}\end{cases}$$

condition (13) will be violated. The resulting contradiction proves the necessity of condition (12) for the pair $\left(\hat{\chi}(\cdot),\hat{\psi}_0(\cdot)\right)$ to be a solution of problem (11).

Sufficiency. Let the pair $\left(\hat{\chi}(\cdot),\hat{\psi}_0(\cdot)\right)$ satisfy condition (12) almost everywhere for $x\in\Omega$. We will show that it is a solution to the problem (12). Let $x\in\Omega$ $\chi(x)\in\Lambda_x,\psi_0(x)\in\Phi_x$. Then, almost everywhere for $x\in\Omega$

$$Q\left(\hat{\chi}(x),\hat{\psi}_0(x)\right) \le Q\left(\chi(x),\hat{\psi}_0(x)\right),$$
$$Q\left(\hat{\chi}(x),\hat{\psi}_0(x)\right) \ge Q\left(\hat{\chi}(x),\psi_0(x)\right).$$

Integrating these inequalities over all $x\in\Omega$ and taking into account that the inequality may fail only on a set of points in Ω, where the values of the integrand do not affect the value of the integral, we obtain:

$$\int_\Omega Q\left(\hat{\chi}(x),\hat{\psi}_0(x)\right)dx \le \int_\Omega Q\left(\chi(x),\hat{\psi}_0(x)\right)dx,$$
$$\int_\Omega Q\left(\hat{\chi}(x),\hat{\psi}_0(x)\right)dx \ge \int_\Omega Q\left(\hat{\chi}(x),\psi_0(x)\right)dx.$$

Adding the constant $-\sum_{i=1}^{N}\overline{\psi}_i b_i$ to both sides of the obtained inequalities, we obtain inequalities (13) and (14). Theorem 2 is proven.

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Theorem 3. The optimal solution of Problem **B** is determined by the following formulas: for all $l = 1,\dots,L$, and almost everywhere for $x \in \Omega$

$$\hat{\chi}_l(x) = \begin{cases} 1, \text{ if } \sum_{i=1}^{N}\left(\left(c(x,\hat{\tau}_i)/w_i + a_i\right)/k + \gamma_i^l \hat{\psi}_i\right)\lambda_i^l = \\ = \min_{s=1,L} \sum_{i=1}^{N}\left(\left(c(x,\hat{\tau}_i)/w_i + a_i\right)/k + \gamma_i^s \hat{\psi}_i\right)\lambda_i^s, \\ 0 \quad \text{otherwise}, \qquad \forall l = \overline{1,L}; \end{cases} \tag{15}$$

where ($\hat{\tau}_1,\dots,\hat{\tau}_N$, $\hat{\psi}_1,\dots,\hat{\psi}_N$) is the solution to the following problem:

$$G(\boldsymbol{\psi}) = \min_{\boldsymbol{\tau}^N \in \hat{\Omega}^N} G_1(\boldsymbol{\tau},\boldsymbol{\psi}) \to \max \tag{16}$$

subject to the conditions:

$$\psi_i \ge 0, \quad i = p+1,\dots,N, \tag{17}$$

where

$$G_1\left(\boldsymbol{\tau}^N,\boldsymbol{\psi}\right) = -\sum_{i=1}^{N}\psi_i b_i +$$
$$+\int_\Omega \min_{\substack{\sigma_l \in M(N_s,k) \\ l=\overline{1,L}}} \sum_{i\in\sigma_l}\left[\left(c(x,\tau_i)/w_i + a_i\right)/k + \gamma_i^l \psi_i\right]\rho(x)dx.$$

Proof. The reduction of Problem **B** to Problem **C** was justified above. Theorem 1 reduces the solution of the latter to finding a saddle point of its Lagrange functional. Fix the vectors $\overline{\boldsymbol{\psi}} \in \Psi$ and $\overline{\boldsymbol{\tau}}^N \in \hat{\Omega}^N$ in it. According to Theorem 2, to determine the remaining components of the saddle point of the Lagrange functional, it is necessary to solve problem (12) for each point x from Ω.

Let x be an arbitrarily fixed point from Ω. Due to the separability of the function $Q(\chi(x),\psi_0(x))$ with respect to its variables, the following equality holds:

$$\max_{\psi_0(x)\in\Phi_x} \min_{\chi(x)\in\Lambda_x} Q(\chi(x),\psi_0(x)) =$$
$$= \max_{\psi_0\in\Phi_x} \min_{\chi\in\Lambda_x} \sum_{l=1}^{L}\left(\sum_{i=1}^{N}(d_i + \gamma_i^l\overline{\psi}_i)\lambda_i^l\rho + \psi_0\right)\chi_l - \psi_0 =$$
$$= \max_{\psi_0^l}\left\{\sum_{l=1}^{L}\min_{0\le\chi_l\le1}\left[\sum_{i=1}^{N}\left(d_i(x) + \gamma_i^l\overline{\psi}_i\right)\lambda_i^l\rho(x) + \psi_0\right]\chi_l - \psi_0\right\}.$$

The point $(\hat{\chi}(x),\hat{\psi}_0(x))$ will be a solution to the problem (12) if and only if the following conditions [16] are satisfied:

1. $Q(\hat{\chi}(x),\hat{\psi}_0(x)) = \min_{\chi(x)\in\Lambda_x} Q(\chi(x),\hat{\psi}_0(x))$;
2. $\dfrac{\partial Q(\hat{\chi}(x),\hat{\psi}_0(x))}{\partial\psi_0} = 0 \iff \sum_{l=1}^{L}\chi_l(x) - 1 = 0$.

For an arbitrary $\psi_0(x)$, the function $Q(\chi(x),\psi_0(x))$ attains its minimum value over all $\chi(x)\in\Lambda_x$, where $\Lambda_x = \{\chi = (\chi_1,\dots,\chi_l,\dots,\chi_L) : 0 \le \chi_l \le 1, l = 1,\dots,L\}$. At the point $\hat{\chi}(x)$, whose components satisfy the conditions: for $l = 1,\dots,L$

$$\hat{\chi}_l(x) = \begin{cases} 1, \text{ if } \sum_{i=1}^{N}\left(d_i(x) + \gamma_i^l\overline{\psi}_i\right)\lambda_i^l\rho(x) + \psi_0(x) < 0, \\ 0, \text{ if } \sum_{i=1}^{N}\left(d_i(x) + \gamma_i^l\overline{\psi}_i\right)\lambda_i^l\rho(x) + \psi_0(x) > 0, \\ \alpha, \text{if } \sum_{i=1}^{N}\left(d_i(x) + \gamma_i^l\overline{\psi}_i\right)\lambda_i^l\rho(x) + \psi_0(x) = 0, \end{cases} \tag{18}$$

where $\alpha \in [0,1]$.

Among all solutions of Problem **C**, we consider those that are extreme points of the feasible set of its solutions. Due to the arbitrariness of the choice of $\alpha \in [0,1]$ we consider a particular case of formula (18), namely: for $l = 1,\dots,L$,

$$\hat{\chi}_l(x) = \begin{cases} 1, \quad \text{if } \sum_{i=1}^{N}\left(d_i(x) + \gamma_i^l\overline{\psi}_i\right)\lambda_i^l\rho(x) + \psi_0(x) < 0, \\ 0, \quad \text{if } \sum_{i=1}^{N}\left(d_i(x) + \gamma_i^l\overline{\psi}_i\right)\lambda_i^l\rho(x) + \psi_0(x) > 0, \\ 0 \vee 1, \text{if } \sum_{i=1}^{N}\left(d_i(x) + \gamma_i^l\overline{\psi}_i\right)\lambda_i^l\rho(x) + \psi_0(x) = 0. \end{cases} \tag{19}$$

For $\psi_0(x) = \hat{\psi}_0(x)$ in inequality (18), particularly (19), it holds the equality: $\sum_{l=1}^{N}\chi_l(x) - 1 = 0$. Then among the components of the vector $\hat{\chi}(x)$ in (19), there is only one unit component, let its index be *l*, and this formula can be written as:

$$\hat{\chi}_l(x) = \begin{cases} 1, \text{ if } \sum_{i=1}^{N}\left(d_i(x) + \gamma_i^l\overline{\psi}_i\right)\lambda_i^l\rho(x) + \hat{\psi}_0(x) = \\ = \min_{s=1,L}\sum_{i=1}^{N}\left(d_i(x) + \gamma_i^s\overline{\psi}_i\right)\lambda_i^s\rho(x) + \hat{\psi}_0(x), \\ 0 \quad \text{otherwise}, \qquad \forall l = \overline{1,L}. \end{cases} \tag{20}$$

Substituting (20) into the function $Q(\chi(x),\psi_0(x))$, we obtain:

$$Q(\hat{\chi}(x),\hat{\psi}_0(x)) = \min_{s=1,L}\left(\sum_{i=1}^{N}(d_i(x) + \gamma_i^s\overline{\psi}_i)\lambda_i^s\rho(x) + \hat{\psi}_0(x)\right) -$$
$$-\hat{\psi}_0(x) = \min_{s=1,L}\left(\sum_{i=1}^{N}\left(d_i(x) + \gamma_i^s\overline{\psi}_i\right)\lambda_i^s\right)\rho(x).$$

Due to the arbitrariness of the choice of point x, the optimal value of the functional in problem (11) with fixed vectors $\overline{\boldsymbol{\psi}} \in \Psi$ and $\overline{\boldsymbol{\tau}}^N \in \hat{\Omega}^N$ is expressed as:

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$$G\left(\overline{\boldsymbol{\tau}}^N,\overline{\boldsymbol{\psi}}\right)=W\left(\left(\hat{\chi}(\cdot),\overline{\boldsymbol{\tau}}^N\right),\left(\hat{\psi}_0(\cdot),\overline{\boldsymbol{\psi}}\right)\right)=$$

$$=\int_\Omega \min_{s=\overline{1,L}}\left(\sum_{i=1}^N\left(d_i(x)+\gamma_i^s\overline{\psi}_i\right)\lambda_i^s\right)\rho(x)dx-\sum_{i=1}^N\psi_i b_i ,$$

or, substituting the expression for $d_i(x)$ and avoiding the use of index indicators that form the combination σ_s, $s=\overline{1,L}$:

$$G(\overline{\boldsymbol{\tau}}^N,\overline{\boldsymbol{\psi}})=-\sum_{i=1}^N\psi_i b_i+$$

$$+\int_\Omega \min_{\substack{\sigma_l\in M(N_s,k)\\ l=1,\dots,L}}\sum_{i\in\sigma_l}((c(x,\overline{\tau}_i)/w_i+a_i)/k+\gamma_i^l\overline{\psi}_i)\rho(x)dx.$$

Due to the arbitrariness of the choice of vectors $\overline{\boldsymbol{\psi}}$ and $\overline{\boldsymbol{\tau}}^N$, and taking into account the obtained expression, we rewrite the functional of problem (10), excluding the function $\psi_0(x)$, in the following form:

$$\overline{H}(\boldsymbol{\psi})=-\sum_{i=1}^N\psi_i b_i+$$

$$+\min_{\tau^N\in\hat{\Omega}^N}\int_\Omega \min_{\substack{\sigma_l\in M(\mathbf{N},k)\\ l=1,\dots,L}}\sum_{i\in\sigma_l}((\frac{c(x,\tau_i)}{w_i}+a_i)/k+\gamma_i^l\psi_i)\rho(x)dx.$$

According to (19), almost everywhere for $x\in\Omega_{\sigma_l}$ the following system of inequalities holds:

$$\begin{cases}\sum_{i=1}^N\left(d_i(x)+\gamma_i^l\overline{\psi}_i\right)\lambda_i^l\rho(x)+\psi_0(x)\le 0,\\ -\sum_{i=1}^N\left(d_i(x)+\gamma_i^s\overline{\psi}_i\right)\lambda_i^s\rho(x)-\psi_0(x)\le 0,\forall s=\overline{1,L},s\ne l.\end{cases}$$

This system is solvable due to the solvability of Problem **C** (and **B**). Adding each of the remaining inequalities to the l-th inequality, we obtain almost everywhere for $x\in\Omega_{\sigma_l}$, $\forall s=\overline{1,L},s\ne l$

$$\sum_{i=1}^N\left(d_i(x)+\gamma_i^l\overline{\psi}_i\right)\lambda_i^l\rho(x)\le\sum_{i=1}^N\left(d_i(x)+\gamma_i^s\overline{\psi}_i\right)\lambda_i^s\rho(x).$$

Recalling that $\rho(x)\ge 0$ for $x\in\Omega$, almost everywhere for $x\in\Omega_{\sigma_l}$ the following system of inequalities will hold: $\forall s=\overline{1,L},\ s\ne l$

$$\sum_{i=1}^N\left(d_i(x)+\gamma_i^l\overline{\psi}_i\right)\lambda_i^l\le\sum_{i=1}^N\left(d_i(x)+\gamma_i^s\overline{\psi}_i\right)\lambda_i^s.$$

Formula (19), considering the above, can be written as follows:

$$\hat{\chi}_l(x)=\begin{cases}1,\text{if }\sum_{i=1}^N(d_i(x)+\gamma_i^l\overline{\psi}_i)\lambda_i^l=\min_{s=\overline{1,L}}\sum_{i=1}^N(d_i(x)+\gamma_i^s\overline{\psi}_i)\lambda_i^s,\\ 0\quad\text{otherwise},\qquad\forall l=\overline{1,L}.\end{cases}$$

For $\hat{\boldsymbol{\psi}}\in\Psi$ and $\hat{\boldsymbol{\tau}}^N\in\hat{\Omega}^N$ to be components of the saddle point of the Lagrange functional of Problem **C**, they must be the optimal solution to the following problem:

$$G(\boldsymbol{\psi})=\min_{\boldsymbol{\tau}^N\in\hat{\Omega}^N}G_1(\boldsymbol{\tau},\boldsymbol{\psi})\to\max,$$

$$\psi_i\ge 0,\quad i=p+1,\dots,N,$$

where

$$G_1(\boldsymbol{\tau}^N,\boldsymbol{\psi})=-\sum_{i=1}^N\psi_i b_i+$$

$$+\int_\Omega \min_{\substack{\sigma_l\in M(\mathbf{N},k)\\ l=1,\dots,L}}\sum_{i\in\sigma_l}((\frac{c(x,\tau_i)}{w_i}+a_i)/k+\gamma_i^l\psi_i)\rho(x)dx.$$

Thus, the characteristic functions of the subsets $\Omega^*_{\sigma_l}, l=\overline{1,L}$, which form the optimal multiplex partition, are found using the following formula: for all $l=1,\dots,L$, and almost everywhere $x\in\Omega$

$$\hat{\chi}_l(x)=\begin{cases}1,\ \text{if }\sum_{i=1}^N((c(x,\hat{\tau}_i)/w_i+a_i)/k+\gamma_i^l\hat{\psi}_i)\lambda_i^l=\\ =\min_{s=\overline{1,L}}\sum_{i=1}^N((c(x,\hat{\tau}_i)/w_i+a_i)/k+\gamma_i^s\hat{\psi}_i)\lambda_i^s,\\ 0\quad\text{otherwise},\qquad\forall l=\overline{1,L};\end{cases}$$

where ($\hat{\tau}_1,...,\hat{\tau}_N$, $\hat{\psi}_1,...,\hat{\psi}_N$) is the solution to the problem:

$$G(\boldsymbol{\psi})=\min_{\boldsymbol{\tau}^N\in\hat{\Omega}^N}G_1(\boldsymbol{\tau},\boldsymbol{\psi})\to\max,\ \psi_i\ge 0,\ i=\overline{p+1,N}.$$

Theorem 3 is proven.

Remark. Let $\gamma_j^l=\frac{1}{k}$ for all $j=\overline{1,N}$ and $l=\overline{1,L}$. Then the indices in these parameters can be omitted, and almost everywhere for $x\in\Omega$ in (15), the component $\hat{\chi}_l(x)=1$ when

$$\sum_{i=1}^N\frac{1}{k}(c(x,\hat{\tau}_i)/w_i+a_i+\hat{\psi}_i)\lambda_i^l=$$

$$=\min_{s=\overline{1,L}}\frac{1}{k}\sum_{i=1}^N(c(x,\hat{\tau}_i)/w_i+a_i+\hat{\psi}_i)\lambda_i^s.$$

Considering that for any $s=\overline{1,L}$ among the N values λ_i^s only k are nonzero, the minimum value of the sum on the right will be achieved at the index l, such that

$$\forall i\in\sigma_l,\forall j\in\mathbf{N}\setminus\sigma_l:(\frac{c(x,\hat{\tau}_i)}{w_i}+a_i)+\hat{\psi}_i\le(\frac{c(x,\hat{\tau}_j)}{w_j}+a_j)+\hat{\psi}_j.$$

That is $\hat{\chi}_l(x)=1$ corresponds to the vector $\hat{\lambda}^l$ with the following components:

$$\hat{\lambda}_i^l(x)=\begin{cases}1,\text{if }\frac{c(x,\hat{\tau}_i)}{w_i}+a_i+\hat{\psi}_i\le\frac{c(x,\hat{\tau}_j)}{w_j}+a_j+\hat{\psi}_j,\\ \qquad\forall i\in\sigma_l,\ j\in\mathrm{N}\setminus\sigma_l,\\ 0\ \text{otherwise}.\end{cases}$$

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Thus, when the demand for services in the Ω_{σ_l}, $l=\overline{1,L}$, is distributed evenly among the centers, the mathematical formulation of the multiplex partition problem can be limited to the characteristic vector-function $\lambda(\cdot)=(\lambda_1(\cdot),...,\lambda_N(\cdot))$, defined on Ω by the following rule: if $x\in\Omega_{\sigma_l}$, then $\lambda_i(x)=1 \forall i\in\sigma_l$, and $\lambda_j(x)=1\forall j\in\mathbf{N}\setminus\sigma_l$, $l=\overline{1,L}$. Then, in terms of characteristic functions, Problem **A** is written as follows:

Problem D:

$$\min_{(\lambda(\cdot),\boldsymbol{\tau}^{N-m})\in\hat{\Gamma}^k\times\hat{\Omega}^{N-m}} I\left(\lambda(\cdot),\boldsymbol{\tau}^{N-m}\right),$$

where:

$$I(\lambda(\cdot),\boldsymbol{\tau}^{N-m})=\frac{1}{k}\int_\Omega\sum_{i=1}^{N}\left(c(x,\tau_i)/w_i+a_i\right)\lambda_i(x)\rho(x)dx,$$

$$\hat{\Gamma}^k=\left\{\lambda(\cdot):\lambda(\cdot)\in\hat{\Gamma}_0^k\right.,$$

$$\frac{1}{k}\int_\Omega\rho(x)\lambda_i(x)dx=b_i,\quad i=1,\ldots,p,$$

$$\left.\frac{1}{k}\int_\Omega\rho(x)\lambda_i(x)dx\le b_i,\quad i=p+1,\ldots,N\right\};$$

$$\hat{\Gamma}_0^k=\left\{\lambda(\cdot)=(\lambda_1(\cdot),...,\lambda_N(\cdot)):\lambda_i(x)=0\vee 1,\, i=1,\ldots,N,\right.$$

$$\left.\sum_{i=1}^{N}\lambda_i(x)=k,\, l=1,\ldots,L,\text{ almost everywhere for } x\in\Omega\right\}.$$

The optimal solution to this problem, based on the above material, can also be expressed as follows:

$$\hat{\lambda}_i(x)=\begin{cases}1 if\ \ \forall i\in\sigma(x),\\ 0\ j\in \mathrm{N}\setminus\sigma(x),\ i=\overline{1,N},\end{cases}$$

where $\sigma(x)=\left\{j_1,j_2,\ldots,j_k\right\}$ is the set of indices of the first k elements in the array $D_{sorted}(x)=\left\{\hat{d}_{j_1}(x),\hat{d}_{j_2}(x),\ldots,\hat{d}_{j_N}(x)\right\}$ sorted in ascending order, with elements $\hat{d}_i(x)=\dfrac{c(x,\hat{\tau}_i)}{w_i}+a_i+\hat{\psi}_i$, $i=\overline{1,N}$, and $\hat{\tau}_1,...,\hat{\tau}_N$, $\hat{\psi}_1,...,\hat{\psi}_N$ is the solution to the problem:

$$G(\boldsymbol{\psi})=\min_{\boldsymbol{\tau}^N\in\hat{\Omega}^N}G_1(\boldsymbol{\tau},\boldsymbol{\psi})\to\max,\ \psi_i\ge 0,\ i=\overline{p+1,N},$$

$$G_1(\boldsymbol{\tau}^N,\boldsymbol{\psi})=\frac{1}{k}\int_\Omega\sum_{i\in\sigma(x)}\left(\frac{c(x,\tau_i)}{w_i}+a_i+\psi_i\right)\rho(x)dx-\sum_{i=1}^{N}\psi_i b_i.$$

4 EXPERIMENTS

Based on the formulas obtained in Theorem 3 and the remark to it, computational methods and algorithms for solving continuous problems of optimal multiplex partitioning of sets have been developed, some variants of which are presented in the works [12, 13]. Computational experiments were conducted to verify the correctness of the algorithms and the adequacy of the mathematical model of optimal location of service centers and multiplex allocation of service demand, continuously spread over a certain territory. The latter can be uniform or proportional to the capacities of the centers.

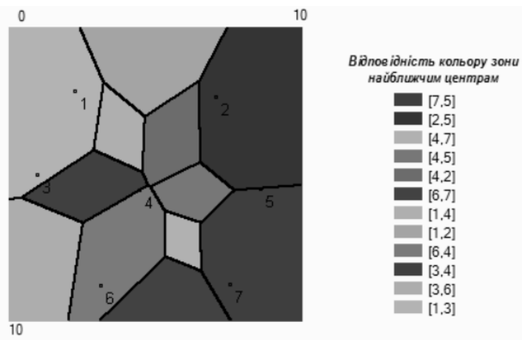
Specific cases of problem A were solved: 1) optimal multiplex partitioning of a set with fixed centers without restrictions on their capacities; 2) OMPS with fixed centers without restrictions; 3) OMPS with location of centers with unlimited and limited their capabilities.

To illustrate the work of the proposed mathematical model and approaches, we developed a software implementation using C# in Visual Studio. For the experimental environment, we used a Lenovo laptop featuring an 8-core Intel Core i7 CPU, 16 GB of RAM, a 512 GB SSD, and running Windows 10.

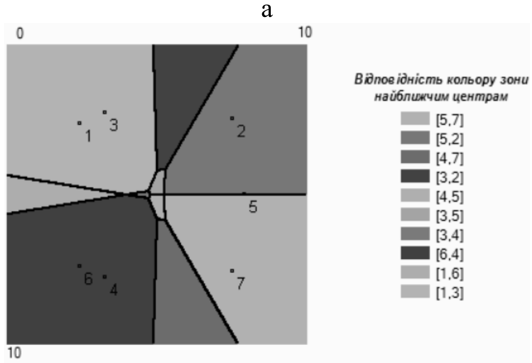
5 RESULTS

In the problems presented below, the following data are common: $\Omega=\left\{x\in R^2:0\le x_i\le 10,\ i=1,2\right\}$; $\rho(x)=1$ $\forall x\in\Omega$; $a_i=0,\ w_i=1,\forall i=\overline{1,N}$; the distance function is the Minkowski metric with parameter p: $c\left(x,\tau_i\right)=\sqrt[p]{(x_1-\tau_1^i)^p+(x_2-\tau_2^i)^p}$.

Problem 1. Figure 1 shows a duplex ($k=2$) partition of the square for $N=7$ fixed (a) and optimally located (b) centers.



a



b

Figure 1 – Duplex partition of the square for 7 centers: a – fixed centers, b – optimally located centers

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The case considered is $p = 2$, with unlimited center capacities, and demand for services is evenly distributed in shared areas. On the right side of the partition in the figures, here and further, the color of each zone corresponds to the pair of centers that must serve it. Table 1 provides the corresponding coordinates of the centers and their calculated capacities.

Since the demand for services is evenly distributed over the entire area and equals 1, the computed capacities essentially represent the area that each center must cover. The objective function value in the OMPS problem with fixed centers is: F = 230.2844, execution time = 2 sec., number of obtained subsets (zones) = 12. The same parameters in the OMPS problem with center location are:
F = 210.6106, execution time = 32 sec, number of zones = 10. As seen, due to the optimal placement of centers, the objective function value decreased by 8.54%.

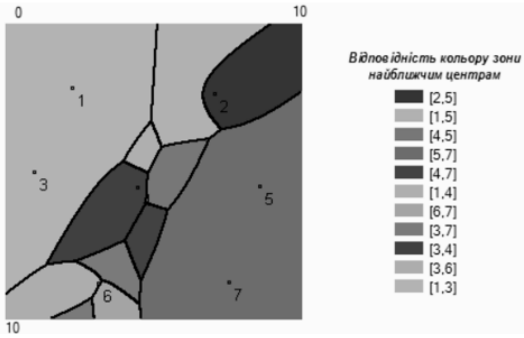
Problem 2. Figure 2 shows a 2nd-order partition of the square for the same fixed centers as in Problem 1, but with capacity constraints (see Table 2).

Table 1 – Coordinates and capacities of centers in Problem 1

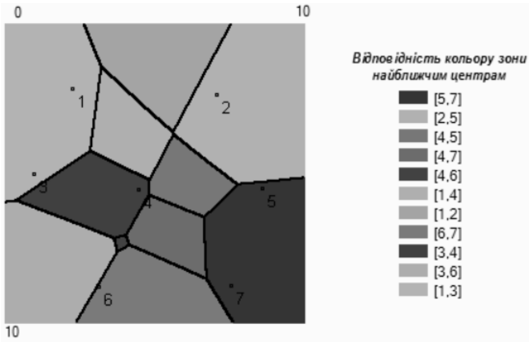
Center №	Center coordinates				Center capacity, b_i	
	fixed		optimally located		fixed	optimally located
	τ_1^i	τ_2^i	τ_{1*}^i	τ_{2*}^i		
1	2.24	2.16	2.389	2.607	13.844	12.813
2	7.04	2.36	7.471	2.453	16.010	12.426
3	0.96	5	3.221	2.236	14.553	14.966
4	4.44	5.52	3.222	7.724	14.408	14.969
5	8.56	5.48	7.899	4.979	17.477	18.789
6	3.12	8.76	2.389	7.352	12.063	12.812
7	7.52	8.72	7.471	7.506	10.847	12.426

The problem was solved when the demand for services in the shared area of two centers is distributed both evenly (Problem D, Fig. 2a) and proportionally (Problem A, Fig. 2b). The number of non-empty zones was the same in both cases: Number of zones = 11, the solution time was almost the same, about 50 seconds, and the objective function values of the direct problem were 322.34 and 296.896, respectively.

As seen from Table 2 and Figure 2b, in the case when demand in the shared area of two centers is distributed proportionally to their capacities, even low-capacity centers do not fully exhaust their capabilities, although they cover a large area. This is explained by the fact that in these areas, most of the work is taken on by centers whose capacities are significantly larger than others. If the problem of optimal duplex partitioning is solved for the same seven fixed centers with the capacities computed above but changing the form of demand distribution in shared areas, the resulting partition is shown in Figure 3.



a



b

Figure 2 – Duplex service zones in Problem 2 for seven centers. Distribution of customers between centers on their common area is: a – uniform, b – proportional

Table 2 – Load on centers in Problem 2

Center №	Capacity b_i	Real capacity calculated in case of distribution customs between centers on their common area	
		uniform	Proportional
1	100	20.773	21.645
2	4	4.062	1.085
3	100	22.204	26.415
4	6	5.941	1.125
5	100	25.083	30.550
6	3	2.988	0.433
7	100	18.149	17.949

The values of the dual (computed and provided to verify the correctness of the algorithm) and direct problem functionals, solution time, number of zones, and calculated capacities of the centers are provided below according to the figure:

Fig. 3,a – F_{dual} = 421.4912, F_{direct}= 421.4927, time = 53 sec., Number zones = 8; B_{real} = (21.645; 1.085; 26.415; 1.125; 30.550; 0.433; 17.949);

Fig. 3,b – F_{dual} = 293.8833, F_{direct}= 293.8835, time = 58 sec., Number zones = 12: B_{real} = (20.773; 4.062; 22.204; 5.941; 25.083; 2.988; 18.150).

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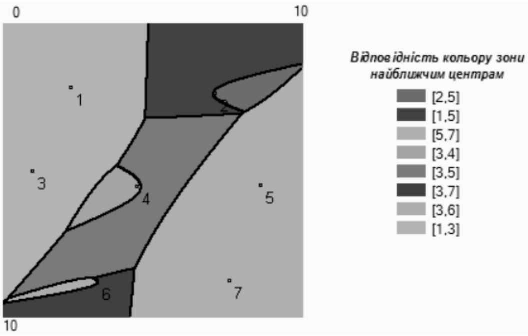
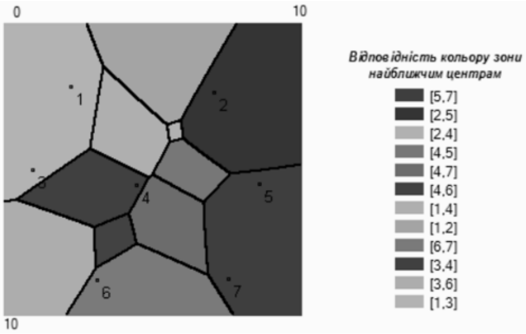


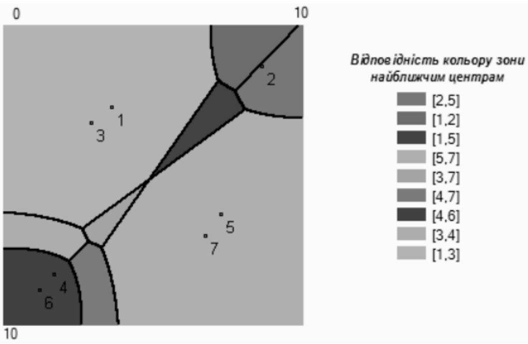
a



b

Figure 3 – Optimal duplex partition of the square for seven centers with limited capacities. Demand in shared zones is distributed: a – uniform (capacity corresponds to Fig. 2 b), b – proportional (capacity like Fig. 2 a)

Problem 3. Figure 4 a shows the solution to the optimal duplex partition problem of the square with the location of seven centers.



a

	Tx	Ty	wi	ai	Bi	realBi
1	3,594...	2,731...	1	0	30	21,974200
2	8,595...	1,372...	1	0	4	4,0025999
3	2,915...	3,245...	1	0	30	20,949400
4	1,672...	8,294...	1	0	6	5,9873999
5	7,250...	6,276...	1	0	30	22,323200
6	1,207...	8,787...	1	0	3	2,9661999
7	6,731...	7,018...	1	0	30	20,998600

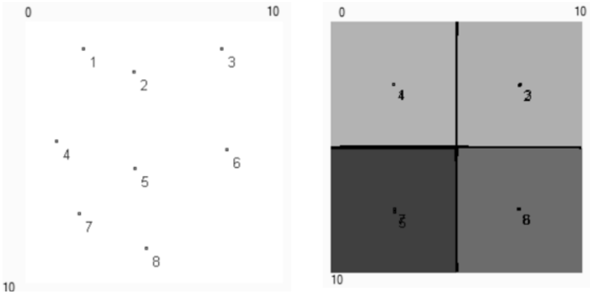
b

Figure 4 – Solution of the optimal duplex partition problem of the square with the location of seven centers with limited capacities: a – partition; b – coordinates of the centers and their capacities

Some of them have limited capacities (in Figure 4b, the grid table shows the coordinates of the placed centers, their limited capacities B_i, and the demand they must cover $realB_i$). The 2nd-order partition of the square for the same fixed centers as in Problem 1, but with capacity constraints (see Table 2), was solved when the demand for services in the shared area of two centers is distributed evenly. The number of non-empty zones is 9, the solution time is 56 seconds, and the objective function value of the direct problem is 277.76.

We present a few more examples in which solutions of problems are predictable and confirm the correctness of the algorithm.

Problem 4. The problem of optimal duplex partitioning with the location of 8 centers. The Minkowski metric parameter is given. The initial placement of centers is shown in Fig. 5, a; the optimally located centers and the corresponding duplex partition are presented in Fig. 5,b. In Fig. 6, the table provides the coordinates of the placed centers, initial parameter values w_i, a_i, b_i, center capacities calculated according to the obtained partition (last column). At the optimal solution, F_{dual} = 166.602, F_{direct} = 166.678.



a b

Figure 5 – Optimal duplex partition of the square with the location of eight centers: a – initial placement of centers; b – optimal placement of centers and their corresponding zones

	Tx	Ty	wi	ai	Bi	realBi
1	2,489...	2,480...	1	0	100	12,498
2	7,473...	2,510...	1	0	100	12,400
3	7,501...	2,484...	1	0	100	12,381
4	2,494...	2,464...	1	0	100	12,317
5	2,504...	7,541...	1	0	100	12,297
6	7,464...	7,454...	1	0	100	12,386
7	2,530...	7,442...	1	0	100	12,499
8	7,435...	7,452...	1	0	100	12,418

Figure 6 – Calculation results in Problem 5

The results of computational experiments for solving optimal triplex partitioning problems of the square with fixed and optimally located centers are shown in Figures 7–9. The Manhattan metric (Figures 7, 8) and the Euclidean metric (Figure 9) were used to calculate the distance between points.

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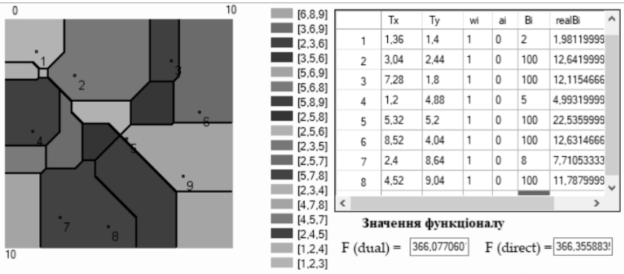

Figure 7 – Optimal triplex partitioning of the square with fixed centers and limited capacities of the 1st, 4th, and 7th centers

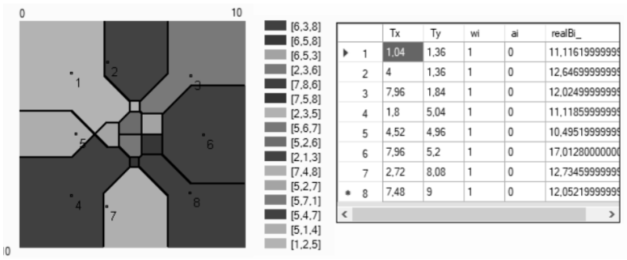

Figure 8 – Optimal triplex partitioning problem of the square with the location of ten centers

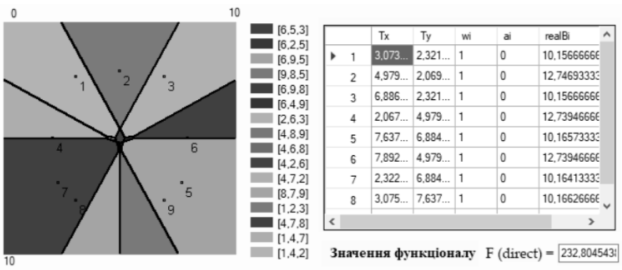

Figure 9 – Optimal triplex partitioning of the square with location of nine centers (Euclidean metric)

6 DISCUSSION

Unlike the OMPS models and problems presented in [4, 5, 12, 13], in this paper, first, the partitioning criterion is refined by considering the average cost of providing a service to a client, calculated for all centers that can serve them. This allows for a more accurate representation of the service provision in the objective function.

Second, the results of multiplex partitioning of sets with constraints on the centers' capacity are presented for different approaches of calculating capacity utilization. Specifically, cases are considered where the demand for a service in the k-th order zone is distributed among the corresponding centers either proportionally to their capacities (as in [12, 13]) or evenly. In both cases, the capacity utilization constraints are met because of optimal partitioning, but the last may differ significantly. Typically, in the first case, the service areas for low-capacity centers are smaller and their capabilities are not fully utilized. The full utilization of the small center capacity is characteristic for an even demand distribution.

Under certain initial conditions, the placement-partitioning results correspond to the properties of the solutions of OMPS problems presented in [12, 13]. When capacity constraints are present, they are always satisfied, and the objective functions of the primal and dual problems converge with acceptable accuracy.

Thus, the obtained results confirm the validity of the developed mathematical model for optimal zoning of territories with the facility location in the form of an optimal multiplex partitioning problem of continuous sets.

CONCLUSIONS

The scientific novelty of this research lies in the theoretical justification of methods and algorithms for optimal multiplex partitioning of sets. This is achieved through the formulation and proof of propositions and theorems that establish the properties of the functional, define the set of feasible solutions, and determine the necessary and sufficient conditions for optimality.

The practical significance of this research lies in the ability to apply the developed methods and algorithms for decision-making in the distribution of structural objects within a logistics system and the determination of their service zones based on specific criteria.

Future research will focus on generalizing these problems by considering the temporal variability of the demand function, the hierarchical structure of logistics systems, and the multi-stage nature of distribution processes. Additionally, efforts will be made to adapt the developed algorithms for practical applications in territorial zoning, taking into account existing infrastructure and interconnections between real-world objects.

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МАТЕМАТИЧНІ ОСНОВИ МЕТОДІВ ОПТИМАЛЬНОГО МУЛЬТИПЛЕКСНОГО РОЗБИТТЯ КОНТИНУАЛЬНИХ МНОЖИН

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АНОТАЦІЯ

Актуальність. Об'єктом дослідження є процес розміщення сервісних центрів (служб соціального захисту, складів аварійного постачання та ін.) і розподілу між ними попиту на послугу у регіоні. Представлено математичні моделі і обґрунтовано методи розв'язання оптимізаційних задач розміщення-розподілу, в яких передбачено перекриття сервісних зон на той випадок, коли найближчий центр не зможе надати послугу. Актуальність дослідження обумовлена необхідністю вирішення завдань, пов'язаних, приміром, з територіальним розподілом об'єктів логістичних систем і завчасним плануванням запобіжних заходів в районах потенційних техногенних аварій, організації евакуаційних процесів або надання первинної гуманітарної допомоги населенню у разі надзвичайних ситуацій.

Мета – забезпечення надання гарантованого сервісу у короткий термін шляхом прикріплення клієнта до декількох найближчих центрів, розроблення відповідного математичного та програмного забезпечення.

Метод. Введено поняття характеристичної вектор-функції розбиття k-го порядку множини, теоретично аргументовано використання процедури ЛП-релаксації задачі, записаної у термінах таких характеристичних функцій. Математичне забезпечення розроблено з використанням елементів функціонального аналізу, теорії двоїстості, негладкої оптимізації.

Результати. Представлено і досліджено математичну модель оптимального територіального зонування з розміщенням центрів, при наявності обмежень на їхні потужності у вигляді неперервної задачі оптимального мультиплексного розбиття множин (ОМРМ), яка описує розподільчі процеси в логістичних системах за критеріями мінімізації відстані до декількох найближчих центрів з урахуванням їх можливостей. Доведено ряд тверджень та теорем стосовно властивостей функціоналу і множини допустимих розв'язків задачі. Отримано необхідні та достатні умови оптимальності, на яких базуються розроблені методи і алгоритми оптимального мультиплексного розбиття множин.

Висновки. Теоретичні положення і результати обчислювальних експериментів, наведені у роботі, свідчать про коректність розробленого математичного апарата і легко переносяться на окремі випадки розглянутої задачі. Доведені твердження та теореми лежать в основі обчислювальних методів оптимального зонування територій із розміщенням центрів, які варто використовувати при організації розподілу матеріальних потоків для оцінювання місткості центрів і парку задіяних транспортних засобів.

КЛЮЧОВІ СЛОВА: континуальна множина, мультиплексне розбиття, оптимізація, ЛП-релаксація, умови оптимальності, задачі розміщення-розподілу.

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