

A METHOD FOR DETERMINING THE FUZZY DISCRETE FRÉCHET DISTANCE

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ABSTRACT

Context. The article addresses the problem of image similarity assessment based on the Fréchet distance metric and its modifications. In this context, images are approximated by polygonal curves. The problem arises from the need to quantitatively evaluate image similarity for tasks such as image generation, clustering, and recognition. Quantitative assessment of the proximity of bio-medical images supports decision-making in automated diagnostic systems. The object of the study is the process of image similarity evaluation. The subject of the study is the Fréchet distance metric and its modifications.

Objective. To develop a method for determining the fuzzy discrete Fréchet distance, to evaluate the computational complexity of the proposed method, to implement the algorithm for determining the fuzzy discrete Fréchet distance in software, and to conduct computational experiments to evaluate the fuzzy discrete Fréchet distance between polygons.

Method. The article presents a method for determining the fuzzy discrete Fréchet distance based on the fuzzy Fréchet metric between polygonal curves. The fuzzy Fréchet metric is grounded in the classical Fréchet distance defined on the space of parameterized curves. The required approximation for practical applications is achieved through the discretization of the fuzzy Fréchet metric. The developed method estimates the fuzzy discrete Fréchet distance between polygonal curves by adapting the algorithm for computing the classical discrete Fréchet distance.

Results. The computer experiments were conducted on a set of predefined regions approximated by polygonal curves. Based on the proposed method, an algorithm was developed to evaluate the discrete fuzzy Fréchet distance. The developed algorithm exhibits low computational complexity, equal to the product of the discretized segments of the polygonal curves: $O(Cm \cdot n)$. This enables the estimation of the discrete Fréchet distance with a specified similarity threshold. The software implementation of the method is intended to be integrated into an automatic medical diagnostic system.

Conclusions. The results obtained in the study allow recommending the developed method for evaluating image similarity based on the fuzzy discrete Fréchet distance for broad application in computer vision systems, including image generation, clustering, and recognition.

KEYWORDS: metric, fuzzy Fréchet metric, fuzzy discrete Fréchet metric, image, polygonal curves.

ABBREVIATIONS

MRI is a Magnetic Resonance Imaging;
RAM is a Random Access Memory;
GB is a gigabyte;
Java is a programming language;
OpenCV is an Open Source Computer Vision Library;
HDD is a Hard Disk Drive;
Windows is a Microsoft's operating system;
Linux is a family of open-source operating systems;
MacOS is an Apple's operating system;
OpenCV 3.0 is a version 3 of the OpenCV library;
Java8+ is a Java version 8 or higher;
intelliJIDEA is an Integrated Development Environment.

NOMENCLATURE

X is a topological space;
 (X, d) is a metric space;
 $C(X)$ is a set of parametric curves in X ;
 $d(x, y)$ is a distance between points x and y ;
 I is a segment $[0, 1]$;
 $H(I)$ is a group of homeomorphisms of the segment I ;
 γ_1 is a parametric curve;
 γ_2 is a parametric curve;

$\sigma(\gamma_1)$ is a polygonal approximations of γ_1 ;
 $\sigma(\gamma_2)$ is a polygonal approximations of γ_2 ;
 $\alpha_{1,2}$ is an arbitrary continuous non-decreasing function on the interval $[0, 1]$, such that $\alpha(0) = 0$ and $\alpha(1) = 1$;
 s is a parameter defined on the interval $[0, 1]$;
 d_F is a discrete Fréchet distance;
 M_d is a fuzzy metric in the sense of George and Veeramani;
 M_F is a fuzzy Fréchet metric;
 Md_F is a discrete fuzzy Fréchet metric;
 $t \in (0; \infty)$ is a threshold parameter for fuzzy distance;
 n is a number of segments of curve γ_1 ;
 m is a number of segments of curve γ_2 ;
 x is an independent variable;
 y is an independent variable;
 z is an independent variable;
 $dC(X)$ is a set of discrete curves in X .
 q is a number of segments of parameter t ;
 p is an index of curve pairs γ_1 and γ_2 ;
 L is a set of node-like (nodal) pairs of curves γ_1 and γ_2 ;
 k is a number of curve pairs γ_1 and γ_2 ;
 i, j are indices of the sequences of points of the pair of curves γ_1 and γ_2 ;

$O_1, O_2 \subseteq \mathbb{R}^2$ are convex regions;
 $\partial O_1, \partial O_2$ are boundaries of regions O_1, O_2 ;
 $O(d_F)$ is a computational complexity of discrete Fréchet distance;
 N is a fuzzy norm.

INTRODUCTION

At the low level, the input is an image. After applying spatial and frequency filtering algorithms [1], as well as edge enhancement algorithms, the output is an enhanced image.

At the middle level, the input is the enhanced image, and the output consists of segmented regions. Edge and texture extraction procedures are applied to these segments. Upon describing the contours and textures, feature vectors are obtained.

At the high level, the input data comprise contour and texture feature vectors. The output of this level is a set of class labels assigned to the input images.

Thus, for each level of computer vision processing, it is necessary to quantitatively evaluate the quality of image transformation algorithms.

Fuzzy metrics are commonly employed for these tasks.

At the low level of image processing, a primary application of fuzzy metrics is noise reduction. Nebojša M. Ralević, Marija Paunović, and Bratislav Iričanin introduced the notions of fuzzy T-metrics and fuzzy S-metrics, and demonstrated their application for image denoising [2].

In another study [3], the authors presented examples of fuzzy metrics in the sense of George and Veeramani, and provided applications of such metrics in color image processing.

In [4], fuzzy wavelets are introduced, which are employed in neural network training.

In [5], the notions of fuzzy T-metric and fuzzy S-metric are connected with image quality indicators, and the proposed approach is used for image filtering.

At the middle level, fuzzy metrics are typically applied for color image comparison, segmentation, and skeletonization of images.

In [6], the notion of fuzzy colors and fuzzy color spaces is introduced and applied to color image comparison. Authors S. Vinurajkumar and S. Anandhavelu applied fuzzy metrics to the segmentation of MRI images [7]. In [8], the notion of a fuzzy skeleton is introduced and applied for image comparison.

The high level of computer vision is concerned with clustering and classification of images based on fuzzy metrics.

In [9], fuzzy clustering procedures are described. In [10], researchers utilize fuzzy metrics to estimate distance measures when aligning a 3D skull model with facial photographs. The authors of [11] propose a novel fuzzy clustering method, Fuzzy C-Ordered-Means, which performs well on noisy data. In [12], an integrated clustering approach called Rough Fuzzy K-Modes is introduced. For

fuzzy cluster optimization, simulated annealing and random forest algorithms are applied. In [13], a fuzzy granulation-degranulation criterion is proposed to evaluate the quality of data separation. The authors of [14] used fuzzy models to construct cascade classifiers that were applied to biomedical image classification.

As the analysis of computer vision levels demonstrates, the problem of image assessment using fuzzy metrics remains highly relevant.

The subject of the research is the process of determining the fuzzy discrete Fréchet distance between images.

The object of the research is fuzzy metrics.

The purpose of the research is to develop a method for computing the fuzzy discrete Fréchet distance.

1 PROBLEM STATEMENT

Let (X, d) be a given metric space. Additionally, suppose that two continuous parametric curves γ_1 and γ_2 are defined.

Then, the Fréchet distance between these curves is defined as follows:

$$d_F(\gamma_1, \gamma_2) = \inf_{t \in I} \sup \{d(\gamma_1(\alpha_1(t)), \gamma_2(\alpha_2(t)))\},$$

where the infimum is taken over all reparameterizations $\alpha_1, \alpha_2: [0, 1] \rightarrow [0, 1]$.

Let the two continuous parametric curves γ_1 and γ_2 be approximated by piecewise linear curves. Then we obtain polygonal curves defined as:

$$\sigma(\gamma_1) = \{\gamma_1(1), \gamma_1(2), \dots, \gamma_1(n)\}, \\ \sigma(\gamma_2) = \{\gamma_2(1), \gamma_2(2), \dots, \gamma_2(m)\}.$$

In [15], an analogue of the Fréchet distance in fuzzy metric spaces was introduced. That paper considers both monotonic and non-monotonic cases of the fuzzy Fréchet distance $Md(x, y, t)$. In [16], an algorithm was developed for computing the discrete Fréchet distance d_F . In [17], the notion of the discrete fuzzy Fréchet metric Md_F was introduced. That study proved that the Fréchet distance between piecewise linear curves in fuzzy normed spaces is equal to the discrete Fréchet distance between discrete curves formed by their vertices.

Therefore, having the definition of the discrete fuzzy Fréchet distance Md_F , it is necessary to develop a method for computing this distance between polygonal curves.

2 REVIEW OF THE LITERATURE

The authors of present paper have developed metrics for evaluating image quality at the middle level of computer vision.

In [18], the application of classical metrics such as the Hausdorff distance, the Fréchet distance, and their modifications – Gromov-Fréchet [19] and Gromov-Hausdorff distances – was analyzed in the context of image processing tasks. The papers [20, 21] demonstrate applications of the Fréchet distance between trees to image segmentation problems.

A number of studies are dedicated to theoretical aspects of fuzzy metrics. For example, [22] presents an alternative approach to the notion of fuzzy metric spaces. Dušan Repovš, Aleksandr Savchenko, and Mykhailo Zarichnyi introduced a fuzzy metric on the space of probability measures, known as the fuzzy Prokhorov metric [23]. In [24], the authors describe a Hausdorff-type fuzzy metric space defined over the set of non-empty closed subsets. This approach has found applications in computer science.

Several publications are focused on improving algorithms for computing the discrete Fréchet distance. In [25], Omrit Filtse proposed a universal approximate simplification for polygonal curves. Bringmann and Mulzer [26] introduced an α -approximation algorithm for each $\alpha \in [1, n]$ with time complexity $O(n \log n + n^2/\alpha)$. In [27], a more efficient α -approximation algorithm was developed with lower computational complexity. Tim Wylie and Binhai Zhu [28] studied the subset chain matching problem for polygonal curves based on the discrete Fréchet distance and proved that three variants of this problem are NP-complete. In [29], the notion of a mean curve was introduced, consisting of the vertices of input curves. The authors aimed to minimize the maximum discrete Fréchet distance to all input curves. In [30], an improved algorithm was proposed for computing the Fréchet distance between closed curves. In [31], the authors developed a subquadratic algorithm for computing the discrete Fréchet distance between two sequences of points in the plane.

The analysis of the literature confirms that the discrete Fréchet distance is widely used in computer vision to measure the distance between images. However, a significant research gap remains in the development of a fuzzy variant of the discrete Fréchet distance.

3 MATERIALS AND METHODS

Let X be an arbitrary set. A function $d: X \times X \rightarrow R$ is called a metric on X if, for all $x, y, z \in X$, the following conditions are satisfied [32]:

- 1) $d(x, y) \geq 0$ (non-negativity);
- 2) $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles);
- 3) $d(x, y) = d(y, x)$ (symmetry);
- 4) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

If only conditions (2), (3), and (4) are satisfied, the function is called a pseudometric. If only conditions (1), (2), and (4) are satisfied, the function is called a quasimetric.

Let (X, d) be a metric space. Let $C(X)$ denote the set of parametric curves $\gamma: [0, 1] \rightarrow X$ in the space X .

Define $H(I)$ as the set of non-decreasing continuous self-maps of I that preserve the endpoints.

For two given continuous parametric curves $\gamma_i: I \rightarrow X_i, i = 1, 2$, the Fréchet distance is defined as:

$$d_F(\gamma_1, \gamma_2) = \inf \sup_{t \in I} \{d(\gamma_1(\alpha_1(t)), \gamma_2(\alpha_2(t)))\},$$

where the infimum is taken over all reparameterizations $\alpha_1, \alpha_2 \in H(I)$.

It is known that the function d_F is a metric on the set $C(X)$.

In [17], the notion of a fuzzy Fréchet metric was introduced.

The notion of a fuzzy metric uses the notion of a t -norm, i.e. an associative, commutative, monotone, continuous operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$, for which 1 is a neutral element [33].

A function $M: X \times X \times (0; \infty) \rightarrow [0, 1]$ is called a fuzzy metric in the sense of George and Veeramani if the following conditions hold:

- 1) $M(x, y, t) > 0$;
- 2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
- 3) $M(x, y, t) = M(y, x, t)$;
- 4) $M(x, y, t) * M(y, z, t) \leq M(x, z, t + s)$;
- 5) the function $M(x, y, -): [0, \infty) \rightarrow [0, 1]$ is continuous.

Given a metric d on the set X , define the function $M_d: X \times X \times (0; \infty) \rightarrow [0, 1]$:

$$M_d(x, y, t) = \frac{t}{d(x, y) + t}.$$

This function is a fuzzy metric [32].

Let X be a (finite-dimensional) linear space R^n . A fuzzy norm on X is a function

$$N: X \times [0, \infty) \rightarrow [0, 1]$$

satisfying the following conditions [34]:

- 1) $N(x, 0) = 0$;
- 2) $N(x, t) \equiv 1$ if and only if $x = 0$;
- 3) $N(\lambda x, t) = N(x, t/|\lambda|)$ for all $t \geq 0, x \in X$ and $\lambda \neq 0$;
- 4) $N(x + y, t + s) \geq \min\{N(x, t), N(y, s)\}$ for all $x, y \in X, t, s \geq 0$;
- 5) $\lim_{t \rightarrow \infty} N(x, t) = 1$.

Each fuzzy norm N induces a fuzzy metric M on X (under the min t -norm) via:

$$M(x, y, t) = N(x - y, t), x, y \in X, t \in [0, \infty).$$

Moreover, any (classical) norm $\|\cdot\|$ on X induces a fuzzy norm $N_{\|\cdot\|}$ on X defined by the formula:

$$N_{\|\cdot\|}(x, t) = \frac{t}{t + \|x\|}, N_{\|\cdot\|}(0, 0) = 0.$$

This fuzzy norm is called the standard fuzzy norm [35].

In addition to the standard fuzzy norm, article [36] considers a class of fuzzy norms generated by quasiconcave functions.

In the context of image recognition, the case $n = 2$ is of particular importance.

Let us consider two convex regions $O_1, O_2 \subset R^2$. It is known that the Hausdorff distance between O_1 and O_2 is

equal to the Fréchet distance between their boundaries $\partial O_1, \partial O_2$ [37]. One of our results is an extension of this equality to the case of fuzzy norms on $\mathbf{R}^2$ and the fuzzy Fréchet distance.

For functions $\gamma_i(I)$, $i = 1, 2$, and for any $\alpha \in H(I)$, we have: $\inf_{s \in I} M(\gamma_1(\alpha(s), \gamma_2(s), t)) > 0$ and $M_F(\gamma_1, \gamma_2, t) > 0$.

Then, the following theorem holds:

Theorem. The function M_F is a fuzzy metric on the set $C(X)$.

Let us now define a discrete curve in a fuzzy metric space (X, M, t) as a mapping $\gamma: \{1, 2, \dots, n\} \rightarrow X$, where $n \in \mathbf{N}$ [33].

For n , ($n \in \mathbf{N}$), define a sequence of index pairs $(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)$ of distinct points satisfying the following conditions:

- 1) $(i_1, j_1) = (1, 1)$;
- 2) $(i_k, j_k) = (n, n)$;
- 3) $i_{p+1} \in \{i_p, i_{p+1}\}$;
- 4) $i_{q+1} \in \{i_q, i_{q+1}\}$.

Let the set of all such sequences be denoted by $C(n, m)$.

For discrete curves $\gamma_i: \{1, 2, \dots, n\} \rightarrow X$, $i = 1, 2$, in a fuzzy metric space, we consider a pair:

$$L \in C(n_1, n_2), L = ((i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)), t \in (0; \infty).$$

Define:

$$\|L, t\| = \min \{M(\gamma_1(i_p), \gamma_2(i_p), t) \mid p \in \{1, 2, \dots, k\}\}.$$

Then, the discrete fuzzy Fréchet distance is defined by the formula: $Md_F(\gamma_1, \gamma_2, t) = \max_{L \in C(n_1, n_2)} \|L(t)\|$, which is

defined over the set $dC(X)$ of discrete curves in X .

Based on the introduced metrics – namely, the fuzzy Fréchet metric and the fuzzy discrete Fréchet metric – a method is developed for computing the fuzzy discrete Fréchet distance between arbitrary polygonal curves.

The method consists of the following steps:

- 1) Given two arbitrary polygonal curves $\gamma_1: [0, n] \rightarrow X$, $\gamma_2: [0, m] \rightarrow X$.

For each polygonal curve, define the corresponding sequences:

$$\begin{aligned} \sigma(\gamma_1) &= \{\gamma_1(1), \gamma_1(2), \dots, \gamma_1(n)\}, \\ \sigma(\gamma_2) &= \{\gamma_2(1), \gamma_2(2), \dots, \gamma_2(m)\}. \end{aligned}$$

These sequences satisfy the conditions (1–4) defined previously.

- 2) Define the parameter t via a sequence:

$$\sigma(t) = \{t_1, t_2, \dots, t_q\}.$$

- 3) Construct the set L :

$$L = ((i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)).$$

- 4) Compute the value:

$$\|L, t\| = \min \{M(\gamma_1(i_p), \gamma_2(i_p), t_q \mid p \in \{1, 2, \dots, k\}\}.$$

- 5) Determine the fuzzy discrete Fréchet distance using the formula:

$$Md_F(\gamma_1, \gamma_2, t) = \max_{L \in C(n_1, n_2)} \|L(t)\|.$$

4 EXPERIMENTS

To conduct computational experiments, a method for computing the fuzzy discrete Fréchet distance was implemented in software.

For this purpose, a software module was developed in Java using the OpenCV library. The minimum system requirements are as follows: RAM – 1 GB; HDD – 100 GB; operating system – Windows, Linux, or macOS; programming language – Java 8 or higher; OpenCV library version – 3.0; development environment – IntelliJ IDEA.

The dataset of polygonal curves consists of a set of grayscale images with a resolution of 174×120 pixels. During preprocessing, threshold segmentation is applied to separate the background from the object. In the subsequent step, contours are extracted and approximated using piecewise linear representations.

The threshold t is implemented as a separate variable and, depending on the computed Fréchet distance, the result is displayed in the console in the appropriate format. The threshold t is a dynamic variable and is entered manually upon request by the program.

The pseudocode for computing the fuzzy discrete Fréchet distance Md_F is presented below:

Function $Md_F(\gamma_1, \gamma_2, t)$: real;

input polygonal sequence

$\sigma(\gamma_1) = \{\gamma_1(1), \gamma_1(2), \dots, \gamma_1(n)\}$;

$\sigma(\gamma_2) = \{\gamma_2(1), \gamma_2(2), \dots, \gamma_2(m)\}$;

$\sigma(t) = \{t_1, t_2, \dots, t_q\}$;

function d_F (computation of the discrete Fréchet distance)

begin

if $fd_F < t_i$, then $Md_F = d_F$

else $Md_F = \infty$

end.

Pseudocode for computing the function d_F is provided in [16].

As stated in [16], the computational complexity of calculating the discrete Fréchet distance is:

$$O(d_F) = O(m \cdot n).$$

When computing the fuzzy discrete Fréchet distance, the parameter t is evaluated, which defines the boundary of the Fréchet distance.

Thus, the computational complexity of calculating the fuzzy discrete Fréchet distance is:

$$O(Md_F) = O(Cm \cdot n),$$

where C is a constant.

5 RESULTS

Based on the developed software module for computing the fuzzy discrete Fréchet distance, computational experiments were conducted between polygonal curves.

Table 1 presents the results of calculating the discrete Fréchet distance between the given polygons.

Table 1 – Polygons and the discrete Fréchet distance between them

Image 1	Image 2	Fréchet Distance
		71.84
		130.86
		55.54
		36.06
		69.23

Table 2 presents the results of calculating the fuzzy discrete Fréchet distance for three specified threshold values: t_1 , t_2 and t_3 .

Table 2 – Fuzzy discrete Fréchet distance for the specified threshold values t_1, t_2, t_3

Experiment No	Fuzzy Fréchet Distance	Parameter t		
		$t_1=40$	$t_2=70$	$t_3=100$
1	71.84	–	–	+
2	117.27	–	–	–
3	130.86	–	–	–
4	55.54	–	+	–
5	36.06	+	–	–
6	67.18	–	+	–
7	69.23	–	+	–
8	46.52	–	+	–
9	46.04	–	+	–
10	50.21	–	+	–
11	69.23	–	+	–
12	46.40	–	+	–

Continuation of Table 1

13	35.22	+	–	–
14	48.79	–	+	–
15	80.95	–	–	+
16	70.57	–	–	+
17	64.14	–	+	–
18	54.0	–	+	–
19	34.18	+	–	–
20	83.59	–	–	+

6 DISCUSSION

The fuzzy Fréchet metric was developed in [15]. In [16], a method and an efficient algorithm for computing the discrete Fréchet distance with low computational complexity were proposed. In the present work, a method for computing the fuzzy discrete Fréchet distance is developed. Based on this method, an algorithm is constructed for calculating the fuzzy discrete Fréchet distance, which builds upon the classical discrete Fréchet algorithm and also maintains low computational complexity.

The evaluation of distance based on the fuzzy discrete Fréchet metric (FDFM) makes it possible to identify objects that are within a specified threshold t .

Analysis of Table 1 shows that the discrete Fréchet distance effectively estimates image similarity based on contours: the smaller the distance value, the more similar the image contours are, and vice versa.

Table 2 presents the values of the fuzzy discrete Fréchet distance for given thresholds t , which essentially represent the fuzzy variant of the discrete Fréchet distance. Distance evaluation based on the fuzzy discrete Fréchet distance enables the ranking and selection of images whose contour distances are less than or equal to a specified threshold t .

CONCLUSIONS

In computer vision, there exists a fundamental problem of estimating image similarity. To address this challenge, various distance metrics are employed. Therefore, the development of new metrics for measuring image similarity remains a relevant and important research problem. Metrics for evaluating image similarity are typically based on classical distances, such as the Fréchet distance and the Hausdorff distance. The Fréchet distance is commonly used to assess the similarity between image contours, whereas the Hausdorff distance is applied to evaluate the proximity between image regions.

The authors have developed a number of metrics for image distance estimation [18–20]. The current study continues the research presented in [18–20] and builds upon the previously proposed fuzzy Fréchet metrics [17, 19].

The developed method and algorithm for computing the fuzzy discrete Fréchet distance allow the estimation of discrete contour distances under a specified threshold. This can be applied to image clustering and classification tasks.

The **scientific novelty** of the study lies in the development of a method for evaluating the fuzzy discrete Fréchet distance.

The **practical significance** consists in the algorithmic formulation and software implementation of the proposed method, as well as in the performance of computer experiments on a dataset of polygonal curves.

Directions for future research include the application of the fuzzy discrete Fréchet distance for image clustering problems, based on the proposed fuzzy discrete metric.

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REFERENCES

- Gonzalez R. C., Woods R. E. Digital Image Processing. Fourth Edition, Pearson Education, 2018, 1020 p.
- Ralević N. M., Paunović M. V., Iričanin B. D. Fuzzy metric space and applications in image processing, *Math. Mon.*, 2020, Vol. 48, pp. 103–117. DOI:10.20948/mathmontis-2020-48-9
- Gregori V., Morillas S., Sapena A. Examples of fuzzy metrics and applications, *Fuzzy Sets and Systems*, 2011, Vol. 170, № 1, pp. 95–111. DOI: 10.1016/j.fss.2010.10.019
- Golestaneh P., Zekri M., Sheikholeslam F. Fuzzy wavelet extreme learning machine, *Fuzzy Sets and Systems*, 2018, Vol. 342, pp. 90–108. DOI: 10.1016/j.fss.2017.12.006
- Ralević N. M., Karaklić D., Pištinić N. Fuzzy metric and its applications in removing the image noise, *Soft Computing*, 2019, Vol. 23, № 22, pp. 12049–12061. DOI: 10.1007/s00500-019-03762-5
- Soto-Hidalgo J. M., Sánchez D., Chamorro-Martínez J., Martínez-Jiménez P. M. Color comparison in fuzzy color spaces, *Fuzzy Sets and Systems*, 2020, Vol. 390, pp. 160–182. DOI: 10.1016/j.fss.2019.09.013
- Vinurajkumar S., Anandhavelu S. An Enhanced Fuzzy Segmentation Framework for extracting white matter from T1-weighted MR images, *Biomedical Signal Processing and Control*, 2022, Vol. 71, P. 103093. DOI: 10.1016/j.bspc.2021.103093
- Bloch I. Fuzzy skeleton and skeleton by influence zones: a review. In *Skeletonization: Theory, Methods and Applications*. Elsevier, Academic Press, 2017, pp. 71–87. DOI: 10.1016/B978-0-08-101291-8.00004-3

9. Soto J., Flores-Sintas A., Palarea-Albaladejo J. Improving probabilities in a fuzzy clustering partition, *Fuzzy Sets and Systems*, 2008, Vol. 159, № 4, pp. 406–421. DOI: 10.1016/j.fss.2007.08.016
10. Campomanes-Álvarez C., Campomanes-Álvarez B. R., Guadarrama S., Ibáñez O., Córdón O. An experimental study on fuzzy distances for skull-face overlay in craniofacial superimposition, *Fuzzy Sets and Systems*, 2017, Vol. 318, pp. 100–119. DOI: 10.1016/j.fss.2016.06.015
11. Leski J.M. Fuzzy c-ordered-means clustering, *Fuzzy Sets and Systems*, 2016, Vol. 286, pp. 114–133. DOI: 10.1016/j.fss.2014.12.007
12. Saha I., Sarkar J. P., Maulik U. Integrated Rough Fuzzy Clustering for Categorical Data Analysis, *Fuzzy Sets and Systems*, 2019, Vol. 361, № 15, pp. 1–32. DOI: 10.1016/j.fss.2018.02.007
13. Bandyopadhyay S., Saha S., Pedrycz W. Use of a fuzzy granulation-degranulation criterion for assessing cluster validity, *Fuzzy Sets and Systems*, 2011, Vol. 170, № 1, pp. 22–42. DOI: 10.1016/j.fss.2010.11.015
14. Das A., Mohapatra S. K., Mohanty M. N. Design of deep ensemble classifier with fuzzy decision method for biomedical image classification, *Applied Soft Computing*, 2022, Vol. 115, P. 108178. DOI: 10.1016/j.asoc.2021.108178
15. Bazylevych L., Berezsky O., Zarichnyi M. Fréchet fuzzy metric, *Matematychni Studii*, 2022, Vol. 57, No. 2, pp. 210–215. DOI: 10.30970/ms.57.2.210-215.
16. Eiter, T.; Mannila, H. Computing Discrete Fréchet Distance. Tech. Report CD-TR 94/64, Christian Doppler Laboratory for Expert Systems, TU Vienna, Austria, 1994.
17. Berezkyi M., Berezsky O., Zarichnyi M. Discrete Fréchet fuzzy metric, *Fuzzy Sets and Systems*, 2025. (submitted).
18. Berezsky O., Zarichnyi M. In: Shakhovska N., Medyk-ovskyy M. O. (eds) Metric Methods in Computer Vision and Pattern Recognition, *Advances in Intelligent Systems and Computing V. CSIT 2020. Advances in Intelligent Systems and Computing*, V. 1293. Springer, Cham. DOI: 10.1007/978-3-030-63270-0_13.
19. Berezsky O., Zarichnyi M. Fréchet distance between weighted rooted trees, *Matematychni Studii*, 2017, Vol. 48, No.2, pp. 165–170. DOI: 10.15330/ms.48.2.165-170
20. Berezsky O., Zarichnyi M. Gromov-Fréchet distance between curves, *Matematychni Studii*, 2018, Vol. 50, No. 1, pp. 88–92.
21. Berezsky O., Zarichnyi M., Pitsun O. Development of a metric and the methods for quantitative estimation of the segmentation of biomedical images, *Eastern-European Journal of Enterprise Technologies*, 2017, Vol. 6, № 4, pp. 4–11. DOI: 10.15587/1729-4061.2017.119493
22. Sostak A. George-Veeramani Fuzzy Metrics Revised, *Axioms*, 2018, Vol. 7, № 3, P. 60. DOI: 10.3390/axioms7030060
23. Repovš D., Savchenko A., Zarichnyi M. Fuzzy Prokhorov metric on the set of probability measures, *Fuzzy Sets and Systems*, 2011, Vol. 175, № 1, pp. 96–104. DOI: 10.1016/j.fss.2011.02.014
24. Gutiérrez García J., Romaguera S., Sanchis M. An identification theorem for the completion of the Hausdorff fuzzy metric, *Fuzzy Sets and Systems*, 2013, Vol. 227, pp. 96–106. DOI: 10.1016/j.fss.2013.04.012
25. Filtser O. Universal Approximate Simplification Under the Discrete Fréchet Distance, *Information Processing Letters*. – 2018, Vol. 132, pp. 22–27. DOI: 10.1016/j.ipl.2017.10.002
26. Bringmann K., Mulzer W. Approximability of the discrete Fréchet distance, *Journal of Computational Geometry*, 2016, Vol. 7, № 2, pp. 46–76. DOI: 10.20382/jocg.v7i2a4
27. Chan T. M., Rahmati Z. An Improved Approximation Algorithm for the Discrete Fréchet Distance, *Information Processing Letters*, 2018, Vol. 138, pp. 72–74. DOI: 10.1016/j.ipl.2018.06.011
28. Wylie T., Zhu B. Following a Curve with the Discrete Fréchet Distance, *Theoretical Computer Science*, 2014, Vol. 556, pp. 34–44. DOI: 10.1016/j.tcs.2014.06.026
29. Ahn H.-K., Alt H., Buchin M., Oh E., Scharf L., Wenk C. Middle Curves Based on Discrete Fréchet Distance, *Computational Geometry*, 2020, Vol. 89, P. 101621. DOI: 10.1016/j.comgeo.2020.101621
30. Vodolazskiy E. Discrete Fréchet Distance for Closed Curves, *Computational Geometry*, 2023, Vol. 111, P. 101967. DOI: 10.1016/j.comgeo.2022.101967
31. Agarwal P.K., Ben Avraham R., Kaplan H., Sharir M. Computing the Discrete Fréchet Distance in Subquadratic Time, *SIAM Journal on Computing*, 2014, Vol. 43, No. 2, pp. 429–449. DOI: 10.1137/130920526
32. Deza M. M., Deza E. Encyclopedia of Distances. Springer-Verlag Berlin Heidelberg, 2013, 583 p.
33. Schweizer B., Sklar A. Statistical metric spaces, *Pacific Journal of Mathematics*, 1960, Vol. 10, pp. 313–334. DOI: 10.2140/pjm.1960.10.313
34. Bag T., Samanta S. K. Finite dimensional fuzzy normed linear spaces, *Journal of Fuzzy Mathematics*, 2003, Vol. 11, pp. 687–705.
35. Font J. J., Galindo J., Macario S., Sanchis M. Mazur-Ulam type theorems for fuzzy normed spaces, *Journal of Nonlinear Sciences and Applications*, 2017, Vol. 10, № 8, pp. 4499–4506. DOI: 10.22436/jnsa.010.08.41
36. Sánchez J. C., González D. M. A natural correspondence between quasiconcave functions and fuzzy norms, *Fuzzy Sets and Systems*, 2023, Vol. 466, P. 108413. DOI: 10.1016/j.fss.2022.10.005
37. Godau M. On the complexity of measuring the similarity between geometric objects in higher dimensions. PhD thesis, Freie Universität Berlin, 1999. URL: <https://doi.org/10.17169/refubium-7780>

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МЕТОД ВИЗНАЧЕННЯ НЕЧІТКОЇ ДИСКРЕТНОЇ ВІДСТАНІ ФРЕШЕ

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АНОТАЦІЯ

Актуальність. У статті розглянуто проблему оцінки подібності зображень на основі метрики Фреше та її модифікацій. При цьому зображення апроксимовані полігональними кривими. Проблема обумовлена необхідністю кількісно оцінювати подібність зображень для задач генерування, кластеризації та розпізнавання зображень. Кількісна оцінка близькості біомедичних зображень допомагає підтримувати прийняття рішення в системах автоматичного діагностування. Об'єктом дослідження є процес оцінки подібності зображень. Предметом дослідження є метрика Фреше та її модифікації.

Мета роботи – розробити метод визначення нечіткої дискретної відстані Фреше, оцінити обчислювальну складність розробленого методу, програмно реалізувати алгоритм методу визначення нечіткої дискретної відстані Фреше та провести комп'ютерні експерименти з визначення нечіткої дискретної відстані Фреше між полігонами.

Метод. У статті розроблено метод визначення дискретної нечіткої відстані Фреше, що базується на нечіткій метриці Фреше між полігональними кривими. В основі нечіткої метрики Фреше лежить метрика Фреше на просторі параметризованих кривих. Потрібна для застосувань апроксимація задається процедурою дискретизації нечіткої метрики Фреше. Розроблений метод оцінює дискретну нечітку відстань Фреше між полігональними кривими на основі алгоритму обчислення дискретної відстані Фреше.

Результати. Комп'ютерні експерименти проводились на наборі заданих областей, апроксимованих полігональними кривими. На основі запропонованого методу розроблено алгоритм оцінки дискретної нечіткої відстані Фреше. Розроблений алгоритм має низьку обчислювальну складність, яка рівна добутку дискретизованих відрізків полігональних кривих: $O(Cm \cdot n)$. Це дало змогу оцінювати дискретну відстань Фреше із заданим порогом подібності. Програмна реалізація методу буде інтегрована в систему автоматичного медичного діагностування.

Висновки. Отримані результати дослідження дозволяють рекомендувати розроблений метод оцінки близькості зображень на основі нечіткої дискретної відстані Фреше в системах комп'ютерного зору для широкого застосування: генерування, кластеризації та розпізнавання зображень.

КЛЮЧОВІ СЛОВА: метрика, нечітка метрика Фреше, дискретна нечітка метрика Фреше, зображення, полігональні криві.

ЛІТЕРАТУРА

1. Gonzalez R. C. Digital Image Processing / R. C. Gonzalez, R. E. Woods. – Fourth Edition, Pearson Education, 2018. – 1020 p.
2. Ralević N. M. Fuzzy metric space and applications in image processing / N. M. Ralević, M. V. Paunović, B. D. Iričanin // Math. Mon. – 2020. – Vol. 48. – P. 103–117. DOI:10.20948/mathmontis-2020-48-9
3. Gregori V. Examples of fuzzy metrics and applications / V. Gregori, S. Morillas, A. Sapena // Fuzzy Sets and Systems. – 2011. – Vol. 170, № 1. – P. 95–111. DOI: 10.1016/j.fss.2010.10.019
4. Golestaneh P. Fuzzy wavelet extreme learning machine / P. Golestaneh, M. Zekri, F. Sheikholeslam // Fuzzy Sets and Systems. – 2018. – Vol. 342. – P. 90–108. DOI: 10.1016/j.fss.2017.12.006
5. Ralević N. M. Fuzzy metric and its applications in removing the image noise / N. M. Ralević, D. Karaklić, N. Pištinjat // Soft Computing – 2019. – Vol. 23, № 22. – P. 12049–12061. DOI: 10.1007/s00500-019-03762-5
6. Color comparison in fuzzy color spaces / [J. M. Soto-Hidalgo, D. Sánchez, J. Chamorro-Martínez, P. M. Martínez-Jiménez] // Fuzzy Sets and Systems. – 2020. – Vol. 390. – P. 160–182. DOI: 10.1016/j.fss.2019.09.013
7. Vinurajkumar S. An Enhanced Fuzzy Segmentation Framework for extracting white matter from T1-weighted MR images / S. Vinurajkumar, S. Anandhavelu // Biomedical Signal Processing and Control. – 2022. – Vol. 71. – P. 103093. DOI: 10.1016/j.bspc.2021.103093
8. Bloch I. Fuzzy skeleton and skeleton by influence zones: a review / I. Bloch. – In Skeletonization: Theory, Methods and Applications. Elsevier – Academic Press, 2017. – P. 71–87. DOI: 10.1016/B978-0-08-101291-8.00004-3
9. Soto J. Improving probabilities in a fuzzy clustering partition / J. Soto, A. Flores-Sintas, J. Palarea-Albaladejo // Fuzzy Sets and Systems. – 2008. – Vol. 159, № 4. – P. 406–421. DOI: 10.1016/j.fss.2007.08.016
10. An experimental study on fuzzy distances for skull–face overlay in craniofacial superimposition / [C. Campomanes-Álvarez, B. R. Campomanes-Álvarez, S. Guadarrama, O. Ibáñez, O. Córdón] // Fuzzy Sets and Systems. – 2017. – Vol. 318. – P. 100–119. DOI: 10.1016/j.fss.2016.06.015
11. Leski J.M. Fuzzy c-ordered-means clustering / J.M. Leski // Fuzzy Sets and Systems. – 2016. – Vol. 286. – P. 114–133. DOI: 10.1016/j.fss.2014.12.007
12. Saha I. Integrated Rough Fuzzy Clustering for Categorical Data Analysis / I. Saha, J. P. Sarkar, U. Maulik // Fuzzy Sets and Systems. – 2019. – Vol. 361, №15. – P. 1–32. DOI: 10.1016/j.fss.2018.02.007
13. Bandyopadhyay S. Use of a fuzzy granulation–degranulation criterion for assessing cluster validity / S. Bandyopadhyay, S. Saha, W. Pedrycz // Fuzzy Sets and Systems. – 2011. – Vol. 170, № 1. – P. 22–42. DOI: 10.1016/j.fss.2010.11.015
14. Das A. Design of deep ensemble classifier with fuzzy decision method for biomedical image classification / A. Das, S. K. Mohapatra, M. N. Mohanty // Applied Soft Comput.

- ing. – 2022. – Vol. 115. – P. 108178. DOI: 10.1016/j.asoc.2021.108178
15. Bazylevych L. Fréchet fuzzy metric / L. Bazylevych, O. Berezsky, M. Zarichnyi // *Matematychni Studii*. – 2022. – Vol. 57, No. 2. – P. 210–215. DOI: 10.30970/ms.57.2.210-215.
16. Eiter T. Computing Discrete Fréchet Distance. Tech. Report CD-TR 94/64, Christian Doppler Laboratory for Expert Systems / T. Eiter, H. Mannila. – TU Vienna, Austria, 1994.
17. Berezkyi M. Discrete Fréchet fuzzy metric / M. Berezkyi, O. Berezsky, M. Zarichnyi // *Fuzzy Sets and Systems*. – 2025. (submitted).
18. Berezsky O. Metric Methods in Computer Vision and Pattern Recognition / O. Berezsky, M. Zarichnyi – In: Shakhovska N., Medykovsky M. O. (eds) // *Advances in Intelligent Systems and Computing V. CSIT 2020. Advances in Intelligent Systems and Computing*. – V. 1293. – Springer, Cham. DOI: 10.1007/978-3-030-63270-0_13.
19. Berezsky O. Fréchet distance between weighted rooted trees / O. Berezsky, M. Zarichnyi // *Matematychni Studii*. – 2017. – Vol. 48, No. 2. – P. 165–170. DOI: 10.15330/ms.48.2.165-170
20. Berezsky O. Gromov-Fréchet distance between curves / O. Berezsky, M. Zarichnyi // *Matematychni Studii*. – 2018. – Vol. 50, No. 1. – P. 88–92.
21. Berezsky O. Development of a metric and the methods for quantitative estimation of the segmentation of biomedical images / O. Berezsky, M. Zarichnyi, O. Pitsun // *Eastern-European Journal of Enterprise Technologies*. – 2017. – Vol. 6, № 4. – P. 4–11. DOI: 10.15587/1729-4061.2017.119493
22. Šostak A. George-Veeramani Fuzzy Metrics Revised / A. Šostak // *Axioms*. – 2018. – Vol. 7, № 3. – P. 60. DOI: 10.3390/axioms7030060
23. Repovš D. Fuzzy Prokhorov metric on the set of probability measures / D. Repovš, A. Savchenko, M. Zarichnyi // *Fuzzy Sets and Systems*. – 2011. – Vol. 175, № 1. – P. 96–104. DOI: 10.1016/j.fss.2011.02.014
24. Gutiérrez García J. An identification theorem for the completion of the Hausdorff fuzzy metric / J. Gutiérrez García, S. Romaguera, M. Sanchis // *Fuzzy Sets and Systems*. – 2013. – Vol. 227. – P. 96–106. DOI: 10.1016/j.fss.2013.04.012
25. Filtser O. Universal Approximate Simplification Under the Discrete Fréchet Distance / O. Filtser // *Information Processing Letters*. – 2018. – Vol. 132. – P. 22–27. DOI: 10.1016/j.ipl.2017.10.002
26. Bringmann K. Approximability of the discrete Fréchet distance / K. Bringmann, W. Mulzer // *Journal of Computational Geometry*. – 2016. – Vol. 7, № 2. – P. 46–76. DOI: 10.20382/jocg.v7i2a4
27. Chan T. M. An Improved Approximation Algorithm for the Discrete Fréchet Distance / T. M. Chan, Z. Rahmati // *Information Processing Letters*. – 2018. – Vol. 138. – P. 72–74. DOI: 10.1016/j.ipl.2018.06.011
28. Wylie T. Following a Curve with the Discrete Fréchet Distance / T. Wylie, B. Zhu // *Theoretical Computer Science*. – 2014. – Vol. 556. – P. 34–44. DOI: 10.1016/j.tcs.2014.06.026
29. Middle Curves Based on Discrete Fréchet Distance / [H.-K. Ahn, H. Alt, M. Buchin et al.] // *Computational Geometry*. – 2020. – Vol. 89. – P. 101621. DOI: 10.1016/j.comgeo.2020.101621
30. Vodolazskiy E. Discrete Fréchet Distance for Closed Curves / E. Vodolazskiy // *Computational Geometry*. – 2023. – Vol. 111. – P. 101967. DOI: 10.1016/j.comgeo.2022.101967
31. Computing the Discrete Fréchet Distance in Subquadratic Time / [P. K. Agarwal, R. Ben Avraham, H. Kaplan, M. Sharir] // *SIAM Journal on Computing*. – 2014. – Vol. 43, No. 2. – P. 429–449. DOI: 10.1137/130920526
32. Deza M. M. Encyclopedia of Distances / M. M. Deza, E. Deza. – Springer-Verlag Berlin Heidelberg, 2013. – 583 p.
33. Schweizer B. Statistical metric spaces / B. Schweizer, A. Sklar // *Pacific Journal of Mathematics*. – 1960. – Vol. 10. – P. 313–334. DOI: 10.2140/pjm.1960.10.313
34. Bag T. Finite dimensional fuzzy normed linear spaces / T. Bag, S. K. Samanta // *Journal of Fuzzy Mathematics*. – 2003. – Vol. 11. – P. 687–705.
35. Mazur-Ulam type theorems for fuzzy normed spaces / [J. J. Font, J. Galindo, S. Macario, M. Sanchis] // *Journal of Nonlinear Sciences and Applications*. – 2017. – Vol. 10, № 8. – P. 4499–4506. DOI: 10.22436/jnsa.010.08.41
36. Sánchez J. C. A natural correspondence between quasiconcave functions and fuzzy norms / J. C. Sánchez, D. M. González // *Fuzzy Sets and Systems*. – 2023. – Vol. 466. – P. 108413. DOI: 10.1016/j.fss.2022.10.005
37. Godau M. On the complexity of measuring the similarity between geometric objects in higher dimensions / M. Godau. – PhD thesis, Freie Universität Berlin, 1999. URL: <https://doi.org/10.17169/refubium-7780>