UDC 519.6

INFLUENCE OF SMOOTHNESS AND DISCRETISATION PARAMETERS ON THE ACCURACY OF NUMERICAL INTEGRATION OF TWO-DIMENSIONAL HIGHLY OSCILLATORY FUNCTIONS

Nechuiviter O. P. – Dr. Sc., Professor, Head of the Department of Information Computer Technologies and Mathematics, Educational and Research Institute "Ukrainian Engineering and Pedagogical Academy" of V. N. Karazin Kharkiv National University, Kharkiv, Ukraine. ROR: https://ror.org/03ftejk10. ORCID: https://orcid.org/0000-0003-2775-8471.

Khurdei Y. L. – Senior Lecturer, Department of Information Computer Technologies and Mathematics, Educational and Research Institute "Ukrainian Engineering and Pedagogical Academy" of V. N. Karazin Kharkiv National University, Kharkiv, Ukraine. ROR : https://ror.org/03ftejk10. ORCID: https://orcid.org/0000-0001-8317-8194.

Ivanov V. V. – Post-graduate student at the Department of Information Computer Technology and Mathematics, Educational and Research Institute "Ukrainian Engineering and Pedagogical Academy" of V. N. Karazin Kharkiv National University, Kharkiv, Ukraine. ROR : https://ror.org/03ftejk10. https://orcid.org/0009-0003-5379-9370

Shnitsar A. S. – Post-graduate student at the Department of Information Computer Technology and Mathematics, Educational and Research Institute "Ukrainian Engineering and Pedagogical Academy" of V. N. Karazin Kharkiv National University, Kharkiv, Ukraine. ROR : https://ror.org/03ftejk10. ORCID: https://orcid.org/0009-0008-4834-6971.

Hishchak O. R. – Post-graduate student at the Department of Information Computer Technology and Mathematics, Educational and Research Institute "Ukrainian Engineering and Pedagogical Academy" of V. N. Karazin Kharkiv National University, Kharkiv, Ukraine. ROR : https://ror.org/03ftejk10. ORCID: https://orcid.org/0009-0002-9362-3647.

Zabornyi A. V. – Post-graduate student at the Department of Information Computer Technology and Mathematics, Educational and Research Institute "Ukrainian Engineering and Pedagogical Academy" of V. N. Karazin Kharkiv National University, Kharkiv, Ukraine. ROR : https://ror.org/03ftejk10. ORCID: https://orcid.org/0009-0006-7026-8215.

Letuta A. A. – Post-graduate student at the Department of Information Computer Technology and Mathematics, Educational and Research Institute "Ukrainian Engineering and Pedagogical Academy" of V. N. Karazin Kharkiv National University, Kharkiv, Ukraine. ROR : https://ror.org/03ftejk10. ORCID: https://orcid.org/0009-0001-4353-4534.

ABSTRACT

Context. Numerical integration of rapidly oscillating functions of several variables is a key concept in engineering models and digital image processing. Despite the availability of various integration methods, the influence of smoothness and discretisation parameters on the accuracy of approximation remains insufficiently studied.

Objective. The aim of this study is to analyze a cubature formula that uses economical interpolation schemes and to systematically investigate the influence of smoothness and discretisation parameters on the accuracy of numerical integration.

Method. There are methods for numerical integration of rapidly oscillating functions of several variables, which are developed using information operators that restore intermediate values of functions based on known values of the function at points, lines, and planes. Such information operators include the operators of O. M. Lytvyn, on the basis of which economical schemes for interpolating functions of several variables have been created. Their application in constructing cubature formulas for approximate calculation of double integrals of rapidly oscillating functions of several variables of general form allows calculations to be performed with high accuracy. The main focus is on the question of how the estimation of the error of numerical integration of two-dimensional rapidly oscillating functions in general form improves with the increase in the smoothness of the function.

Results. The cubature formula of the approximate calculation of the double integral from the rapidly oscillating function of a general form is researched.

Conclusions. A comparative analysis of the accuracy of the cubature formula for different classes of functions showed that the class of differentiability of a function is a determining factor that influences the rate of decrease of the theoretical error of numerical integration. Economical interpolation schemes and a higher level of smoothness of functions provide a significant increase in the accuracy of approximate calculation of integrals of two-dimensional rapidly oscillating functions of general form.

KEYWORDS: digital twin, digital signal image processing, mathematical modeling, numerical integration, cubature formula, highly oscillating functions, interpolation of functions of many variables, sparse grid.

NOMENCLATURE

G is an integration domain;

ℓ_μ is a number of lines into which the region is divided as per variable x and y;

Δ_μ are lengths of the intervals breakdowns of G;

$\tilde{\Delta}_\mu$ are lengths of subintervals breakdowns of G;

f is a function of two variables;

$f(\psi_\mu,\theta)$ is a traces of function on the lines ψ_μ;

$f(\psi,\theta_\nu)$ is a traces of function on the lines θ_ν;

J is an operator that restores the function f on the traces on lines;

$\tilde{J}$ is an operator that restores the function f in the knots;

DOI 10.15588/1607-3274-2026-2-4

J_μ is a single-variable f function restoration operators;

$\tilde{J}_\mu$ is a single-variable f function restoration operators;

$h1_{1\mu}$ are linear splines with respect to the first variable, defined on intervals of length Δ_μ in the operator J;

$H1_{1\mu}$ are linear splines with respect to the second variable, defined on intervals of length Δ_μ in the operator J;

$\tilde{h}1_{1\mu}$ are linear splines with respect to the first variable, defined on intervals of length $\tilde{\Delta}_\mu$ in the operator $\tilde{J}$;

$\tilde{H}1_{1\mu}$ are linear splines with respect to the second variable, defined on intervals of length $\tilde{\Delta}_\mu$ in the operator $\tilde{J}$;

$K_{\mu\nu}$ is a subintegral function for representing f through J;

$\tilde{K}_{\mu\nu}$ is a subintegral function for representing f through $\tilde{J}$;

g is a function of two variables;

$g(\psi_\mu, \theta)$ is a traces of function on the lines ψ_μ;

$g(\psi, \theta_\nu)$ is a traces of function on the lines θ_ν;

O is an operator that restores the function g on the traces on lines;

$\tilde{O}$ is an operator that restores the function g in the knots;

O_μ is a single-variable g function restoration operators;

$\tilde{O}_\mu$ is single-variable g function restoration operators;

$h2_{1\mu}$ are linear splines with respect to the first variable, defined on intervals of length Δ_μ in the operator O;

$H2_{1\mu}$ are linear splines with respect to the second variable, defined on intervals of length Δ_μ in the operator O;

$\tilde{h}2_{1\mu}$ are linear splines with respect to the first variable, defined on intervals of length $\tilde{\Delta}_\mu$ in the operator $\tilde{O}$;

$\tilde{H}2_{1\mu}$ are linear splines with respect to the second variable, defined on intervals of length $\tilde{\Delta}_\mu$ in the operator $\tilde{O}$;

$G_{\mu\nu}$ is a subintegral function for representing g through O;

$\tilde{G}_{\mu\nu}$ is a subintegral function for representing g through $\tilde{O}$;

$H^{2,r}$ is a class of functions whose partial derivatives of order r with respect to the first and the second variables are uniformly bounded by the constant M while the function itself and its mixed partial derivatives of order r are uniformly bounded by the constant $\tilde{M}$;

ω is an oscillation parameter;

I^2 is a two-dimensional integral from highly oscillating functions in a general form;

Φ^2 is a cubature formula for calculating two-dimensional integral of rapidly oscillating functions in general form;

ρ is an error of approximation of the integral by the cubature formula.

INTRODUCTION

Currently, the concept of a digital twin is viewed as the integration of advanced information technologies and is widely used in various areas of life. A digital twin is a virtual copy of its physical counterpart that can process data in real time, monitor the current status, and manage system risks. The basic design of digital twins in the military and medical fields combines a physical counterpart with the latest technologies, such as image processing, mathematical modeling, etc. Digital image processing can analyze visual data, facilitate accurate modeling and monitoring of physical objects.

The Fourier transform is one of the central concepts in mathematical modeling of digital signal and image processing problems. The use of Fourier transforms allows useful information to be separated from noise; when part of an image is distorted or lost, algorithms based on multidimensional Fourier series are used to restore the complete picture. Multidimensional Fourier series are also used to analyze data obtained from satellites, radars, drones or thermal imagers. To construct multidimensional Fourier series, it is necessary to calculate Fourier coefficients. The task boils down to constructing cubature formulas for the approximate calculation of integrals of rapidly oscillating functions of many variables.

It is also important to consider the main problem in calculating multidimensional integrals – the rapid growth of computational costs with an increase in the number of variables. This problem can be avoided by using economical interpolation schemes or sparse grids, which significantly reduce the number of nodes required to achieve a given accuracy compared to classical methods using bilinear interpolation.

Currently, there is a pressing need to generalize and optimize the proposed algorithms and to construct mathematical models where it is necessary to calculate double and triple integrals of rapidly oscillating functions of a general form. i.e. in an irregular case. This task is

DOI 10.15588/1607-3274-2026-2-4

more complex, requires more detailed study and the creation of new approaches to obtain meaningful results.

The object of study is the process of mathematical modeling of digital image processing tasks.

The subject of study is an approximate calculation of the integrals from the rapidly oscillating functions of two variables in a general case.

The aim of the work is to construct the cubature formula for the approximate calculation of the integral of the rapidly oscillating function in a general form in the case when information about functions is specified in the nodes of a two-dimensional O.M. Lytvyn's sparse grid. Particular attention is paid to studying the influence of function smoothness on the accuracy of numerical integration.

1 PROBLEM STATEMENT

It is necessary to construct and investigate the cubature formula of the approximate calculation of the integral of highly oscillating function in a general case

$$I^2(\omega)=\int_0^1\int_0^1 f(x,y)\,e^{i\omega g(x,y)}\,dxdy \qquad (1)$$

when the information about $f(x,y)$ and $g(x,y)$ is specified in the nodes of O.M. Lytvyn's sparse grid. Conduct a comparative analysis of the accuracy of the cubature formula on the class $H^{2,r}\left(M,\tilde{M}\right)$, $r=1,2$.

2 REVIEW OF THE LITERATURE

In medicine and defense, digital twins combine physical systems with mathematical modeling and digital image processing methods, enabling accurate analysis of visual data, adequate reproduction of processes and effective monitoring of complex objects.

The complete structure of a digital twin, its main characteristics, and the process of its existence are described in [1].

In particular, [2] describes a digital twin for breast cancer diagnosis using thermographic images, and [3] describes the structure of digital twins that allow image quality assessment in photoacoustic tomography. It is important to note that [3] uses a Fourier transform-based restoration algorithm and demonstrates the usefulness of digital phantom twins by comparing image restoration quality.

The issue of numerical integration of rapidly oscillating functions is relevant [4–9]. It should be noted that a significant proportion of research is devoted to effective cubature formulas [10–12].

The process of digitization in many areas necessitates the use of innovative mathematical approaches, theories and algorithms. One such innovative approach is the theory of new information operators (O.M. Lytvyn's operators) [13–18], which is based on the construction of operators for restoring functions of several variables using multi-modal data.

Information operators have proven themselves effective in mathematical modeling in various scientific fields, as well as in mathematical modeling of digital signal and image processing tasks. Currently, algorithms have been developed for calculating Fourier coefficients of functions of several variables depending on the type of information about the function. These algorithms, which use function values at points, function traces on lines or planes as data, allow for high accuracy in information processing.

There is already some achievements in this direction of research; namely, some cubature formulas have been constructed for approximate calculation of integrals of rapidly oscillating functions of three and two variables of general form, in the case when the information was given by the values of functions on planes and lines [19, 20]. However, many questions remain unexplored.

These include the creation of appropriate numerical methods for solving engineering modeling problems based on the use of projections, the optimality of the proposed numerical methods, testing of cubature formulas to identify their potential capabilities, and the optimal selection of planes, lines, etc.

It is also important to consider the main problem in calculating multidimensional integrals – the rapid growth of computational costs with an increase in the number of variables. This problem can be avoided by using economical interpolation schemes (O. M. Lytvyn's sparse grid), which significantly reduce the number of nodes required to achieve a given accuracy compared to classical methods using bilinear interpolation.

This article considers the issue of numerical integration of rapidly oscillating functions of two variables in general form when information about the functions is specified at the nodes of a two-dimensional sparse grid. Particular attention is paid to studying the influence of the smoothness of functions on the accuracy of numerical integration.

3 MATERIALS AND METHODS

Let

$$h1_{10}(x)=\begin{cases}0, & x\le x_0,\\ \dfrac{x-x_1}{-\Delta_1}, & x_0<x<x_1,\\ 0, & x\ge x_1,\end{cases}$$

$$h1_k(x)=\begin{cases}0, & x\le x_{k-1},\\ \dfrac{x-x_{k-1}}{\Delta_1}, & x_{k-1}<x<x_k,\\ \dfrac{x-x_{k+1}}{-\Delta_1}, & x_k\le x<x_{k+1},\\ 0, & x\ge x_{k+1},\end{cases} \quad k=\overline{1,\ell_1-1},$$

DOI 10.15588/1607-3274-2026-2-4

$$h1_{1\ell_1}(x)=\begin{cases}0, & x\le x_{\ell_1-1},\\ \dfrac{x-x_{\ell_1}}{\Delta_1}, & x_{\ell_1-1}<x<x_{\ell_1},\\ 0, & x\ge x_{\ell_1},\end{cases}$$

$$H1_{10}(y)=\begin{cases}0, & y\le y_0,\\ \dfrac{y-y_1}{-\Delta_1}, & y_0<y<y_1,\\ 0, & y\ge y_1,\end{cases}$$

$$H1_{1j}(y)=\begin{cases}0, & y\le y_{j-1},\\ \dfrac{y-y_{j-1}}{\Delta_1}, & y_{j-1}<y<y_j,\\ \dfrac{y-y_{j+1}}{-\Delta_1}, & y_j\le y<y_{j+1},\\ 0, & y\ge y_{j+1},\end{cases}\quad j=\overline{1,\ell_1-1},$$

$$H1_{1\ell_1}(y)=\begin{cases}0, & y\le y_{\ell_1-1},\\ \dfrac{y-y_{\ell_1}}{\Delta_1}, & y_{\ell_1-1}<y<y_{\ell_1},\\ 0, & y\ge y_{\ell_1},\end{cases}$$

$$\tilde{h}1_{10}(x)=\begin{cases}0, & x\le \tilde{x}_0,\\ \dfrac{x-\tilde{x}_1}{-\tilde{\Delta}_1}, & \tilde{x}_0<x<\tilde{x}_1,\\ 0, & x\ge \tilde{x}_1,\end{cases}$$

$$\tilde{h}1_{1\tilde{k}}(x)=\begin{cases}0, & x\le \tilde{x}_{\tilde{k}-1},\\ \dfrac{x-\tilde{x}_{\tilde{k}-1}}{\tilde{\Delta}_1}, & \tilde{x}_{\tilde{k}-1}<x<\tilde{x}_{\tilde{k}}\\ \dfrac{x-\tilde{x}_{\tilde{k}+1}}{-\tilde{\Delta}_1}, & \tilde{x}_{\tilde{k}}\le x<\tilde{x}_{\tilde{k}+1},\\ 0, & x\ge \tilde{x}_{\tilde{k}+1},\end{cases}\quad \tilde{k}=\overline{1,\ell_1^2-1},$$

$$\tilde{h}1_{1\ell_1^2}(x)=\begin{cases}0, & x\le \tilde{x}_{\ell_1^2-1},\\ \dfrac{x-\tilde{x}_{\ell_1^2}}{\tilde{\Delta}_1}, & \tilde{x}_{\ell_1^2-1}\le x<\tilde{x}_{\ell_1^2},\\ 0, & x\ge \tilde{x}_{\ell_1^2},\end{cases}$$

$$\tilde{H}1_{10}(y)=\begin{cases}0, & y\le \tilde{y}_0,\\ \dfrac{y-\tilde{y}_1}{-\tilde{\Delta}_1}, & \tilde{y}_0<y<\tilde{y}_1,\\ 0, & y\ge \tilde{y}_1,\end{cases}$$

$$\tilde{H}1_{1\tilde{j}}(y)=\begin{cases}0, & y\le \tilde{y}_{\tilde{j}-1},\\ \dfrac{y-\tilde{y}_{\tilde{j}-1}}{\tilde{\Delta}_1}, & \tilde{y}_{\tilde{j}-1}\le y<\tilde{y}_{\tilde{j}},\\ \dfrac{y-y_{\tilde{j}+1}}{-\tilde{\Delta}_1}, & \tilde{y}_{\tilde{j}}\le y<\tilde{y}_{\tilde{j}+1},\\ 0, & y\ge \tilde{y}_{\tilde{j}+1},\end{cases}\quad \tilde{j}=\overline{1,\ell_1^2-1},$$

$$\tilde{H}1_{1\ell_1^2}(y)=\begin{cases}0, & y\le \tilde{y}_{\ell_1^2-1},\\ \dfrac{y-\tilde{y}_{\ell_1^2}}{\tilde{\Delta}_1}, & \tilde{y}_{\ell_1^2-1}\le y<\tilde{y}_{\ell_1^2},\\ 0, & y\ge \tilde{y}_{\ell_1^2},\end{cases}$$

$$x_k=k\Delta_1,\ y_j=j\Delta_1,\ \Delta_1=\frac{1}{\ell_1};$$

$$\tilde{x}_{\tilde{k}}=\tilde{k}\tilde{\Delta}_1,\ \tilde{y}_{\tilde{j}}=\tilde{j}\tilde{\Delta}_1,\ \tilde{k},\tilde{j}=\overline{0,\ell_1^2},\ \tilde{\Delta}_1=\frac{1}{\ell_1^2}.$$

Auxiliary functions

$$h2_{10}(x), H2_{10}(y),\ h2_{1p}(x),\ p=\overline{1,\ell_2-1},$$

$$H2_{1s}(y),\ s=\overline{1,\ell_2-1},\ h2_{1\ell_2}(x), H2_{1\ell_2}(y),$$

$$\tilde{h}2_{10}(x), \tilde{H}2_{10}(y),\ \tilde{h}2_{1\tilde{p}}(x),\ \tilde{p}=\overline{1,\ell_2^2-1},$$

$$\tilde{H}2_{1\tilde{s}}(y),\ \tilde{s}=\overline{1,\ell_2^2-1},\ \tilde{h}2_{1\ell_2^2}(x),\ \tilde{H}2_{1\ell_2^2}(y).$$

are determined similarly to

$$\tilde{x}_p=\tilde{p}\tilde{\Delta}_2,\ \tilde{y}_s=\tilde{s}\tilde{\Delta}_2,$$

$$\tilde{p},\tilde{s}=\overline{1,\ell_2^2},\ \tilde{\Delta}_2=1/\ell_2^2.$$

Let define operator

$$Jf(x,y)=\sum_{k=0}^{\ell_1} f(x_k,y)h1_{1k}(x)+\sum_{j=0}^{\ell_1} f(x,y_j)H1_{1j}(y)-$$
$$-\sum_{k=0}^{\ell_1}\sum_{j=0}^{\ell_1} f(x_k,y_j)h1_{1k}(x)H1_{1j}(y);$$

$$\tilde{J}f(x,y)=\sum_{k=0}^{\ell_1}\sum_{\tilde{j}=0}^{\ell_1^2} f(x_k,\tilde{y}_{\tilde{j}})h1_{1k}(x)\tilde{H}1_{1\tilde{j}}(y)+$$

$$+\sum_{j=0}^{\ell_1}\sum_{\tilde{k}=0}^{\ell_1^2} f(\tilde{x}_{\tilde{k}},y_j)\tilde{h}1_{1\tilde{k}}(x)H1_{1j}(y)-$$

DOI 10.15588/1607-3274-2026-2-4

$$-\sum_{k=0}^{\ell_1}\sum_{j=0}^{\ell_1} f(x_k, y_j) h1_{1k}(x) H1_{1j}(y);$$

$$Og(x,y)=\sum_{p=0}^{\ell_2} g(x_p, y) h2_{1p}(x)+\sum_{s=0}^{\ell_2} g(x, y_s) H2_{1s}(y)-$$
$$+\sum_{p=0}^{\ell_2}\sum_{s=0}^{\ell_2} g(x_p, y_s) h2_{1p}(x) H2_{1s}(y);$$

$$\tilde{O}g(x,y)=\sum_{p=0}^{\ell_2}\sum_{\tilde{s}=0}^{\ell_2^2} g(x_p, \tilde{y}_{\tilde{s}}) h2_{1p}(x) \tilde{H}2_{1\tilde{s}}(y)+$$
$$+\sum_{s=0}^{\ell_2}\sum_{\tilde{p}=0}^{\ell_2^2} g(\tilde{x}_{\tilde{p}}, y_j) \tilde{h}2_{1\tilde{p}}(x) H2_{1s}(y)-$$
$$-\sum_{p=0}^{\ell_2}\sum_{s=0}^{\ell_2} g(x_p, y_s) h2_{1p}(x) H2_{1s}(y).$$

If additional operators are introduced

$$J_1 f(x,y)=\sum_{k=0}^{\ell_1} f(x_k, y) h1_{1k}(x),$$

$$J_2 f(x,y)=\sum_{j=0}^{\ell_1} f(x, y_j) H1_{1j}(y);$$

$$\tilde{J}_1 f(x,y)=\sum_{\tilde{k}=0}^{\ell_1^2} f(\tilde{x}_{\tilde{k}}, y) \tilde{h}1_{1\tilde{k}}(x),$$

$$\tilde{J}_2 f(x,y)=\sum_{\tilde{j}=0}^{\ell_1^2} f(x, \tilde{y}_{\tilde{j}}) \tilde{H}1_{1\tilde{j}}(y);$$

$$O_1 g(x,y)=\sum_{p=0}^{\ell_2} g(x_p, y) h2_{1p}(x),$$

$$O_2 g(x,y)=\sum_{s=0}^{\ell_2} g(x, y_s) H2_{1s}(y);$$

$$\tilde{O}_1 g(x,y)=\sum_{\tilde{p}=0}^{\ell_2^2} g(\tilde{x}_{\tilde{p}}, y_j) \tilde{h}2_{1\tilde{p}}(x),$$

$$\tilde{O}_2 g(x,y)=\sum_{\tilde{s}=0}^{\ell_2^2} g(x_p, \tilde{y}_{\tilde{s}}) \tilde{H}2_{1\tilde{s}}(y)$$

then for operators $Jf(x,y)$, $Og(x,y)$ and operators $\tilde{J}f(x,y)$, $\tilde{O}g(x,y)$, the following identities will be satisfied:

$$Jf=(J_1+J_2-J_1J_2)f\,,\ Og=(O_1+O_2-O_1O_2)g\,,$$

$$\tilde{J}f=(J_1\tilde{J}_2+\tilde{J}_1J_2-J_1J_2)f\,;$$

$$\tilde{O}g=(O_1\tilde{O}_2+\tilde{O}_1O_2-O_1O_2)g\,.$$

The following cubature formula

$$\Phi^2(\omega)=\int_0^1\int_0^1 \tilde{J}f(x,y)\,\mathrm{e}^{i\omega\tilde{O}g(x,y)}\,dxdy \qquad (2)$$

is proposed for numerical calculation of (1).

Theorem 1 [21]. Let

$$f(x,y),\ g(x,y)\in H^{2,1}(M,\widetilde{M}),$$

then

$$\rho\left(I^2(\omega),\Phi^2(\omega)\right)=$$
$$=\left|\int_0^1\int_0^1 f(x,y)\mathrm{e}^{i\omega g(x,y)}\,dxdy-\int_0^1\int_0^1 \tilde{J}f(x,y)\mathrm{e}^{i\omega\tilde{O}g(x,y)}\,dxdy\right|\le$$
$$\le\frac{\tilde{M}+6M}{9\ell_1^2}+\tilde{M}\min\left(4;\frac{\omega(\tilde{M}+6M)}{9\ell_2^2}\right).$$

Let us consider additional functions that will be used in proving the theorem for representing the approximation error $f(x,y)$ by the operator $Jf(x,y)$ through $f^{(2,2)}(x,y)$:

$$K_{1k}(x,\xi)=\begin{cases}\dfrac{x_{k+1}-x}{x_{k+1}-x_k}(x_k-\xi), & x_k\le\xi<x,\\[2mm] \dfrac{x_k-x}{x_{k+1}-x_k}(x_{k+1}-\xi), & x<\xi\le x_{k+1},\end{cases}$$

$$K_{2j}(y,\eta)=\begin{cases}\dfrac{y_{j+1}-y}{y_{j+1}-y_j}(y_j-\eta), & y_j\le\eta<y,\\[2mm] \dfrac{y_j-y}{y_{j+1}-y_j}(y_{j+1}-\eta), & y<\eta\le y_{j+1},\end{cases}$$

DOI 10.15588/1607-3274-2026-2-4

$$\tilde{K}_{1\tilde{k}}\left(x,\tilde{\xi}\right)=\begin{cases}\dfrac{\tilde{x}_{\tilde{k}+1}-x}{\tilde{x}_{\tilde{k}+1}-\tilde{x}_{\tilde{k}}}\left(\tilde{x}_{\tilde{k}}-\tilde{\xi}\right), & \tilde{x}_{\tilde{k}}\le\tilde{\xi}<x,\\ \dfrac{\tilde{x}_{\tilde{k}}-x}{\tilde{x}_{\tilde{k}+1}-\tilde{x}_{\tilde{k}}}\left(\tilde{x}_{\tilde{k}+1}-\tilde{\xi}\right), & x<\tilde{\xi}\le\tilde{x}_{\tilde{k}+1},\end{cases}$$

$$\tilde{K}_{2\tilde{j}}\left(y,\tilde{\eta}\right)=\begin{cases}\dfrac{\tilde{y}_{\tilde{j}+1}-y}{\tilde{y}_{\tilde{j}+1}-\tilde{y}_{\tilde{j}}}\left(\tilde{y}_{\tilde{j}}-\tilde{\eta}\right), & \tilde{y}_{\tilde{j}}\le\tilde{\eta}<y,\\ \dfrac{\tilde{y}_{\tilde{j}}-y}{\tilde{y}_{\tilde{j}+1}-\tilde{y}_{\tilde{j}}}\left(\tilde{y}_{\tilde{j}+1}-\tilde{\eta}\right), & y<\tilde{\eta}\le\tilde{y}_{\tilde{j}+1}.\end{cases}$$

Lemma. The following is true for function

$$\int_{x_k}^{x_{k+1}}\int_{x_k}^{x_{k+1}}\left|K_{1k}\left(x,\xi\right)\right|d\xi dx=\frac{\Delta_1^{\;3}}{12}.$$

Proof. Note that

$$\int_{x_k}^{x_{k+1}}\left|K_{1k}\left(x,\xi\right)\right|d\xi=$$

$$=\frac{x_{k+1}-x}{x_{k+1}-x_k}\int_{x_k}^{x}\left|x_k-\xi\right|d\xi+\frac{x-x_k}{x_{k+1}-x_k}\int_{x}^{x_{k+1}}\left|x_{k+1}-\xi\right|d\xi=$$

$$=\frac{x_{k+1}-x}{x_{k+1}-x_k}\left(x-x_k\right)+\frac{x-x_k}{x_{k+1}-x_k}\left(x_{k+1}-x\right)$$

then

$$\int_{x_k}^{x_{k+1}}\int_{x_k}^{x_{k+1}}\left|K_{1k}\left(x,\xi\right)\right|d\xi dx=$$

$$=\int_{x_k}^{x_{k+1}}\left(\frac{x_{k+1}-x}{x_{k+1}-x_k}\left(x-x_k\right)+\frac{x-x_k}{x_{k+1}-x_k}\left(x_{k+1}-x\right)\right)dx=$$

$$=\frac{1}{2\Delta}\int_{x_k}^{x_{k+1}}\frac{\left(x-x_k\right)^3}{3}dx+\frac{1}{2\Delta_1}\int_{x_k}^{x_{k+1}}\frac{\left(x_{k+1}-x\right)^3}{3}dx=$$

$$=\frac{1}{2\Delta_1}\left(\frac{\left(x-x_k\right)^4}{4\cdot 3}\Bigg|_{x_k}^{x_{k+1}}-\frac{\left(x_{k+1}-x\right)^4}{4\cdot 3}\Bigg|_{x_k}^{x_{k+1}}\right)=$$

$$=\frac{1}{2\Delta_1}\left(\frac{\Delta_1^{\;4}}{3\cdot 4}+\frac{\Delta_1^{\;4}}{3\cdot 4}\right)=\frac{2\Delta_1^{\;3}}{4!}=\frac{\Delta_1^{\;3}}{12}.$$

Theorem 2. Let $f\left(x,y\right)\in H^{2,2}\left(M,\widetilde{M}\right)$, then

$$\left|\int_0^1\int_0^1 f\left(x,y\right)dxdy-\int_0^1\int_0^1\tilde{J}f\left(x,y\right)dxdy\right|\le\frac{\widetilde{M}+24M}{144\ell_1^{\;4}}.$$

Proof. It is sufficient to show that

$$\left|\int_0^1\int_0^1 f(x,y)dxdy-\int_0^1\int_0^1\tilde{J}f(x,y)dxdy\right|\le$$

$$\le\int_0^1\int_0^1\left|f(x,y)-Jf(x,y)+Jf(x,y)-\tilde{J}f(x,y)\right|dxdy\le$$

$$\le\int_0^1\int_0^1\left|f(x,y)-Jf(x,y)\right|dxdy+$$

$$+\int_0^1\int_0^1\left|Jf(x,y)-\tilde{J}f(x,y)\right|dxdy\le$$

$$\le\int_0^1\int_0^1\left|f(x,y)-Jf(x,y)\right|dxdy+$$

$$+\int_0^1\int_0^1\left|\left(J_1+J_2-J_1J_2\right)f(x,y)-\left(J_1\tilde{J}_2+J_2\tilde{J}_1-J_1J_2\right)f(x,y)\right|dxdy\le$$

$$\le\sum_{k=0}^{\ell_1-1}\sum_{j=0}^{\ell_1-1}\int_{x_k}^{x_{k+1}}\int_{y_j}^{y_{j+1}}\left|\int_{x_k}^{x_{k+1}}\int_{y_j}^{y_{j+1}}f^{(2,2)}(\xi,\eta)K_{1k}\left(x,\xi\right)K_{2j}\left(y,\eta\right)d\xi d\eta\right|dxdy+$$

$$+\sum_{j=0}^{\ell_1-1}\sum_{\tilde{k}=0}^{\ell_1^{\;2}-1}\int_{\tilde{x}_{\tilde{k}}}^{\tilde{x}_{\tilde{k}+1}}\int_{\tilde{x}_{\tilde{k}}}^{\tilde{x}_{\tilde{k}+1}}\left|f^{(2,0)}(\tilde{\xi},y_j)\right|\left|\tilde{K}_{1\tilde{k}}(x,\tilde{\xi})\right|d\tilde{\xi}dx\int_{y_j}^{y_{j+1}}dy+$$

$$+\sum_{k=0}^{\ell_1-1}\sum_{\tilde{j}=0}^{\ell_1^{\;2}-1}\int_{x_k}^{x_{k+1}}dx\int_{\tilde{y}_{\tilde{j}}}^{\tilde{y}_{\tilde{j}+1}}\int_{\tilde{y}_{\tilde{j}}}^{\tilde{y}_{\tilde{j}+1}}\left|f^{(0,2)}(x_k,\tilde{\eta})\right|\left|\tilde{K}_{2_{\tilde{j}}}(y,\tilde{\eta})\right|d\tilde{\eta}dy\le$$

$$\le\widetilde{M}\frac{\Delta_1^{\;2}}{12}\frac{\Delta_1^{\;2}}{12}+2M\ell_1\Delta_1\frac{2\tilde{\Delta}_1^3}{4!}\ell_1^{\;2}=$$

$$=\frac{\widetilde{M}}{144\ell_1^{\;4}}+\frac{M}{6\ell_1^{\;4}}=\frac{\widetilde{M}+24M}{144\ell_1^{\;4}}.$$

Theorem 3. Let $f\left(x,y\right),\ g\left(\mathrm{x},\mathrm{y}\right)\in H^{2,2}\left(M,\tilde{M}\right)$, then

$$\left|\int_0^1\int_0^1 \mathrm{e}^{i\omega g(\mathrm{x},\mathrm{y})}dxdy-\int_0^1\int_0^1 \mathrm{e}^{i\omega\tilde{O}g(\mathrm{x},\mathrm{y})}dxdy\right|\le$$

$$\le\min\left(4;\frac{\omega\left(\tilde{M}+24M\right)}{144\ell_2^{\;4}}\right).$$

Proof. Let us consider additional functions that will be used in proving the theorem for representing the approximation error $g\left(x,y\right)$ by the operator $Of\left(x,y\right)$ through $g^{(2,2)}\left(x,y\right)$:

$$G_{1p}\left(x,t\right)=\begin{cases}\dfrac{x_{p+1}-x}{x_{p+1}-x_p}\left(x_p-t\right), & x_p\le t<x,\\ \dfrac{x_p-x}{x_{p+1}-x_p}\left(x_{p+1}-t\right), & x<t\le x_{p+1},\end{cases}$$

DOI 10.15588/1607-3274-2026-2-4

$$G_{2,s}(y,\tau)=\begin{cases}\dfrac{y_{s+1}-y}{y_{s+1}-y}(y_s-\tau),\ y_s\le\tau<y,\\ \dfrac{y_s-y}{y_{s+1}-y}(y_{s+1}-\tau),\ y<\tau\le y_{s+1}.\end{cases}$$

$$\tilde{G}_{1p}(x,\tilde{t})=\begin{cases}\dfrac{\tilde{x}_{\tilde{p}+1}-x}{\tilde{x}_{\tilde{p}+1}-\tilde{x}_{\tilde{p}}}(\tilde{x}_{\tilde{p}}-\tilde{t}),\ \tilde{x}_{\tilde{p}}\le\tilde{t}<x,\\ \dfrac{\tilde{x}_{\tilde{p}}-x}{\tilde{x}_{\tilde{p}+1}-\tilde{x}_{\tilde{p}}}(\tilde{x}_{\tilde{p}+1}-\tilde{t}),\ x<\tilde{t}\le\tilde{x}_{\tilde{p}+1},\end{cases}$$

$$\tilde{G}_{2,\tilde{s}}(y,\tilde{\tau})=\begin{cases}\dfrac{\tilde{y}_{\tilde{s}+1}-y}{\tilde{y}_{\tilde{s}+1}-\tilde{y}_{\tilde{s}}}(\tilde{y}_{\tilde{s}}-\tilde{\tau}),\ \tilde{y}_{\tilde{s}}\le\tilde{\tau}<y,\\ \dfrac{\tilde{y}_{\tilde{s}}-y}{\tilde{y}_{\tilde{s}+1}-\tilde{y}_{\tilde{s}}}(\tilde{y}_{\tilde{s}+1}-\tilde{\tau}),\ y<\tilde{\tau}\le\tilde{y}_{\tilde{s}+1}.\end{cases}$$

Thus

$$\left|\int_0^1\int_0^1 e^{i\omega g(x,y)}\,dxdy-\int_0^1\int_0^1 e^{i\omega\tilde{O}g(x,y)}\,dxdy\right|\le$$

$$\le\left|\int_0^1\int_0^1 e^{i\omega g(x,y)}\,dxdy-\int_0^1\int_0^1 e^{i\omega Og(x,y)}\,dxdy\right|+$$

$$+\left|\int_0^1\int_0^1 e^{i\omega Og(x,y)}\,dxdy-\int_0^1\int_0^1 e^{i\omega\tilde{O}g(x,y)}\,dxdy\right|\le$$

$$\le\int_0^1\int_0^1\left|e^{i\omega g(x,y)}-e^{i\omega Og(x,y)}\right|dxdy+$$

$$+\int_0^1\int_0^1\left|e^{i\omega Og(x,y)}-e^{i\omega\tilde{O}g(x,y)}\right|dxdy\le$$

$$\le\int_0^1\int_0^1\left|2i\sin\frac{\omega g(x,y)-\omega O g(x,y)}{2}e^{i\frac{\omega}{2}(g(x,y)+Og(x,y))}\right|dxdy+$$

$$+\int_0^1\int_0^1\left|2i\sin\frac{\omega O g(x,y)-\omega\tilde{O}g(x,y)}{2}e^{i\frac{\omega}{2}(Og(x,y)+\tilde{O}g(x,y))}\right|dxdy\le$$

$$\le 2\int_0^1\int_0^1\left|\sin\frac{\omega g(x,y)-\omega O g(x,y)}{2}\right|dxdy+$$

$$+2\int_0^1\int_0^1\left|\sin\frac{\omega O g(x,y)-\omega\tilde{O}g(x,y)}{2}\right|dxdy\le$$

$$\le 2\int_0^1\int_0^1\min\left(1;\frac{\omega\left|g(x,y)-Og(x,y)\right|}{2}\right)dxdy+$$

$$+2\int_0^1\int_0^1\min\left(1;\frac{\omega\left|(O_1+O_2-O_1O_2)g(x,y)-(O_1\tilde{O}_2+O_2\tilde{O}_1-O_1O_2)g(x,y)\right|}{2}\right)dxdy\le$$

$$\le 2\int_0^1\int_0^1\min\left(1;\frac{\omega\left|g(x,y)-Og(x,y)\right|}{2}\right)dxdy+$$

$$+2\int_0^1\int_0^1\min\left(1;\frac{\omega(O_1-O_1\tilde{O}_2)g(x,y)+(O_2-O_2\tilde{O}_1)g(x,y)}{2}\right)dxdy\le$$

$$\le 2\sum_{p=0}^{\ell_2-1}\sum_{s=0}^{\ell_2-1}\int_{x_p}^{x_{p+1}}\int_{y_s}^{y_{s+1}}\min\left(1;\frac{\omega}{2}\left|\int_{x_p}^{x_{p+1}}\int_{y_s}^{y_{s+1}}g^{(2,2)}(t,\tau)G_{1p}(x,t)G_{2s}(y,\tau)\,dtd\eta\right|\right)dxdy+$$

$$+2\min\left(\sum_{\tilde{p}=0}^{\ell_2^2-1}\sum_{\tilde{s}=0}^{\ell_2^2-1}\int_{\tilde{x}_{\tilde{p}}}^{\tilde{x}_{\tilde{p}+1}}\int_{\tilde{y}_{\tilde{s}}}^{\tilde{y}_{\tilde{s}+1}}dxdy;\right.$$

$$\frac{\omega}{2}\sum_{p=0}^{\ell_2-1}\sum_{\tilde{s}=0}^{\ell_2^2-1}\int_{x_p}^{x_{p+1}}dx\int_{\tilde{y}_{\tilde{s}}}^{\tilde{y}_{\tilde{s}+1}}\int_{\tilde{y}_{\tilde{j}}}^{\tilde{y}_{\tilde{j}+1}}\left|g^{(0,2)}(x_p,y)\right|\left|\tilde{G}_{2_{\tilde{s}}}(y,\tilde{\tau})\right|d\tilde{\tau}dy+$$

$$\left.+\frac{\omega}{2}\sum_{s=0}^{\ell_2-1}\sum_{\tilde{p}=0}^{\ell_2^2-1}\int_{x_p}^{x_{p+1}}\int_{\tilde{x}_{\tilde{p}}}^{\tilde{x}_{\tilde{p}+1}}\left|g^{(2,0)}(\tilde{t},y_s)\right|\left|\tilde{G}_{1\tilde{p}}(x,\tilde{t})\right|d\tilde{t}dx\int_{y_s}^{y_{s+1}}dy\right)\le$$

$$\le 2\min\left(\ell_2^2\Delta_2^2,\frac{\tilde{M}\omega}{2}\frac{\Delta_2^2}{12}\frac{\Delta_2^2}{12}\right)+$$

$$+2\min\left(\left(\ell_2^2\Delta_2^2\right)^2,\frac{2M\omega}{2}\ell_2\Delta_2\ell_2^2\frac{2\tilde{\Delta}_2^3}{4!}\right)=$$

$$=\min\left(2;\frac{\tilde{M}\omega}{144\ell_2^4}\right)+\min\left(2;\frac{M\omega}{6\ell_2^4}\right)=$$

$$=\min\left(4;\frac{\omega\left(\tilde{M}+24M\right)}{144\ell_2^4}\right).$$

Theorem 4. Let

$$f(x,y),\ g(x,y)\in H^{2,2}\left(M,\tilde{M}\right),$$

then

$$\rho\left(I^2(\omega),\Phi^2(\omega)\right)=$$

$$=\left|\int_0^1\int_0^1 f(x,y)e^{i\omega g(x,y)}dxdy-\int_0^1\int_0^1\tilde{J}f(x,y)e^{i\omega\tilde{O}g(x,y)}dxdy\right|\le$$

$$\le\frac{\tilde{M}+24M}{144\ell_1^4}+\tilde{M}\min\left(4;\frac{\omega\left(\tilde{M}+24M\right)}{144\ell_2^4}\right).$$

Proof. Let's find the estimate

$$\left|\int_0^1\int_0^1 f(x,y)e^{i\omega g(x,y)}\,dxdy-\int_0^1\int_0^1\tilde{J}f(x,y)e^{i\omega\tilde{O}g(x,y)}\,dxdy\right|\le$$

$$\le\left|\int_0^1\int_0^1 f(x,y)e^{i\omega g\tilde{O}(x,y)}\,dxdy-\int_0^1\int_0^1\tilde{J}f(x,y)e^{i\omega\tilde{O}g(x,y)}\,dxdy\right|+$$

$$+\left|\int_0^1\int_0^1 f(x,y)e^{i\omega g(x,y)}\,dxdy-\int_0^1\int_0^1 f(x,y)e^{i\omega Og(x,y)}\,dxdy\right|+$$

DOI 10.15588/1607-3274-2026-2-4

$$+\left|\int_0^1\int_0^1 f(x,y)e^{i\omega Og(x,y)}dxdy-\int_0^1\int_0^1 f(x,y)e^{i\omega\tilde{O}g(x,y)}dxdy\right|\le$$

$$\le\int_0^1\int_0^1\left|f(x,y)-\tilde{J}f(x,y)\right|\left|e^{i\omega g\tilde{O}(x,y)}\right|dxdy+$$

$$+\int_0^1\int_0^1\left|f(x,y)\right|\left|e^{i\omega g(x,y)}-e^{i\omega Og(x,y)}\right|dxdy+$$

$$+\int_0^1\int_0^1\left|f(x,y)\right|\left|e^{i\omega Og(x,y)}-e^{i\omega\tilde{O}g(x,y)}\right|dxdy\le$$

$$\le\int_0^1\int_0^1\left|f(x,y)-Jf(x,y)+Jf(x,y)-\tilde{J}f(x,y)\right|dxdy+$$

$$+\int_0^1\int_0^1\left|f(x,y)\right|\left|2i\sin\frac{\omega g(x,y)-\omega Og(x,y)}{2}e^{i\frac{\omega}{2}\left(g(x,y)+Og(x,y)\right)}\right|dxdy+$$

$$+\int_0^1\int_0^1\left|f(x,y)\right|\left|2i\sin\frac{\omega Og(x,y)-\omega\tilde{O}g(x,y)}{2}e^{i\frac{\omega}{2}\left(Og(x,y)+\tilde{O}g(x,y)\right)}\right|dxdy\le$$

$$\le\int_0^1\int_0^1\left|f(x,y)-Jf(x,y)\right|dxdy+\int_0^1\int_0^1\left|Jf(x,y)-\tilde{J}f(x,y)\right|dxdy+$$

$$+2\int_0^1\int_0^1\left|f(x,y)\right|\left|\sin\frac{\omega g(x,y)-\omega Og(x,y)}{2}\right|dxdy+$$

$$+2\int_0^1\int_0^1\left|f(x,y)\right|\left|\sin\frac{\omega Og(x,y)-\omega\tilde{O}g(x,y)}{2}\right|dxdy\le$$

$$\le\int_0^1\int_0^1\left|f(x,y)-\left(J_1+J_2-J_1J_2\right)f(x,y)\right|dxdy+$$

$$+\int_0^1\int_0^1\left|\left(J_1+J_2-J_1J_2\right)f(x,y)-\left(J_1\tilde{J}_2+J_2\tilde{J}_1-J_1J_2\right)f(x,y)\right|dxdy+$$

$$+2\tilde{M}\int_0^1\int_0^1\min\left(1;\frac{\omega\left|g(x,y)-\left(O_1+O_2-O_1O_2\right)g(x,y)\right|}{2}\right)dxdy+$$

$$+2\tilde{M}\int_0^1\int_0^1\min\left(1;\frac{\omega\left|\left(O_1+O_2-O_1O_2\right)g(x,y)-\left(O_1\tilde{O}_2+O_2\tilde{O}_1-O_1O_2\right)g(x,y)\right|}{2}\right)dxdy\le$$

$$\le\frac{\tilde{M}+24M}{144\ell_1^4}+\tilde{M}\min\left(4;\frac{\omega\left(\tilde{M}+24M\right)}{144\ell_2^4}\right).$$

Note. The cubature formula (2) uses

$$N_1=2(\ell_1^2+1)(\ell_1+1)-(\ell_1+1)(\ell_1+1)=O(\ell_1^3)$$

values for function $f(x,y)$ and

$$N_2=2(\ell_2^2+1)(\ell_2+1)-(\ell_2+1)(\ell_2+1)=O(\ell_2^3)$$

for function $g(x,y)$.

4 EXPERIMENTS

The essence of the experiment is to compare the accuracy of the cubature formula on the class $H^{2,1}\left(M,\tilde{M}\right)$ and $H^{2,2}\left(M,\tilde{M}\right)$.

To demonstrate the effectiveness of numerical integration of rapidly oscillating functions of two variables of general form using the cubature formula based on O.M. Lytvyn's sparse grid, we will use the estimates obtained in Theorem 1 and Theorem 4. That is, we will show how the accuracy of the approximation significantly depends on the smoothness class of the functions.

Consider the functions $\sin(x+y)$, $\cos(x+y)$, which belong to the class $H^{2,r}\left(M,\tilde{M}\right)$, $r=1,2$, $M=\tilde{M}=1$.

For the given functions, let us consider the behavior of the error of the cubature formula (2). According to Theorem 1, the accuracy of the approximation of the integral (1) by formula (2) on the class $H^{2,1}\left(M,\tilde{M}\right)$ is equal to

$$\varepsilon_1=\varepsilon_1(\omega)=\frac{7}{9\ell^2}+\min\left(4;\frac{7\omega}{9\ell^2}\right),$$

when $\ell_2=\ell$, and according to Theorem 4, on the class $H^{2,2}\left(M,\tilde{M}\right)$ it is equal to

$$\varepsilon_2=\varepsilon_2(\omega)=\frac{25}{144\ell^4}+\min\left(4;\frac{25\omega}{144\ell^4}\right),$$

when $\ell_2=\ell$.

5 RESULTS

Table 1 shows the values of the theoretical errors ε_1 and ε_2 on the class $H^{2,1}\left(M,\tilde{M}\right)$ and $H^{2,2}\left(M,\tilde{M}\right)$, respectively, and for different values of ω and ℓ.

Table 1 provides the values of theoretical errors for two classes of functions with different values of discretisation parameters and different smoothness of functions. As can be seen from the table, the accuracy of numerical integration largely depends on the smoothness of the function: smoother functions provide smaller errors.

The Fig. 1 and Fig. 2 show the curves of theoretical trajectories ε_1 and ε_2 for different values of ω and ℓ.

DOI 10.15588/1607-3274-2026-2-4

Table 1 – Comparison of theoretical errors in $\omega = 20\pi, 60\pi, 100\pi$

ℓ	ω	ε_1	ε_2
64	20π	$1.21 \cdot 10^{-2}$	$6.60 \cdot 10^{-7}$
128	20π	$3.03 \cdot 10^{-3}$	$4.12 \cdot 10^{-8}$
256	20π	$7.58 \cdot 10^{-4}$	$2.58 \cdot 10^{-9}$
512	20π	$1.89 \cdot 10^{-4}$	$1.61 \cdot 10^{-10}$
1024	20π	$4.73 \cdot 10^{-5}$	$1{,}01 \cdot 10^{-12}$
256	60π	$2.24 \cdot 10^{-3}$	$7.66 \cdot 10^{-9}$
512	60π	$5.62 \cdot 10^{-4}$	$4.78 \cdot 10^{-10}$
1024	60π	$1.41 \cdot 10^{-4}$	$2.99 \cdot 10^{-11}$
512	100π	$9.35 \cdot 10^{-4}$	$7.96 \cdot 10^{-10}$
1024	100π	$2.33 \cdot 10^{-4}$	$4.97 \cdot 10^{-11}$

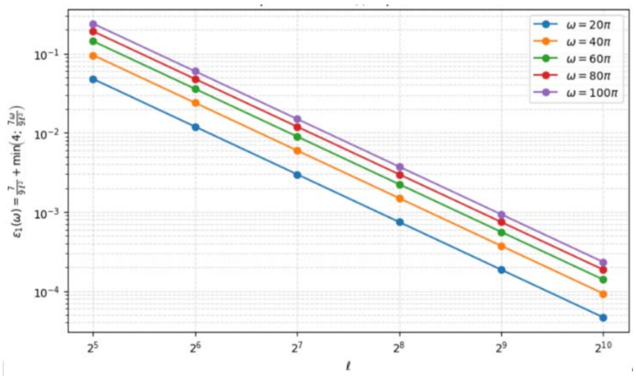

Figure 1 – Curves of theoretical trajectories ε_1 for different values of ω and ℓ

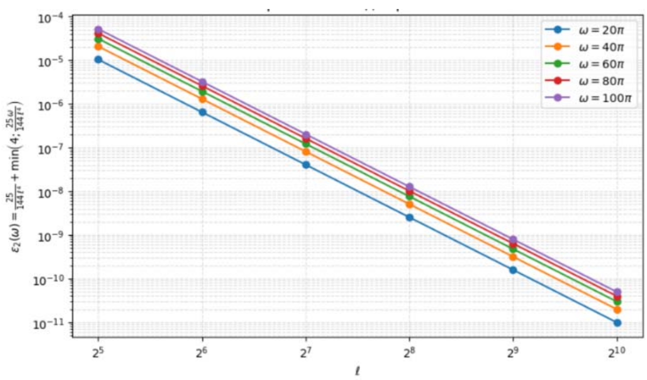

Figure 2 – Curves of theoretical trajectories ε_2 for different values of ω and ℓ

The graphs demonstrate that the accuracy of numerical integration directly depends on the discretisation parameters and the smoothness of the functions.

6 DISCUSSION

The smoothness of functions is a parameter that affects the accuracy when investigating the potential capability of the cubature formula for numerical integration of two-dimensional rapidly oscillating functions of general form. Theoretical estimates of approximation error depend on the class of differentiability: smoother functions provide significantly smaller errors, while less smooth functions are more difficult to approximate.

For the proposed cubature formula, approximation error estimates were obtained for different classes of differentiability. To confirm the theoretical results, theoretical errors were calculated for functions $f(x, y) = \sin(x + y)$, $g(x, y) = \cos(x + y)$, that simultaneously belong to the

DOI 10.15588/1607-3274-2026-2-4

two classes $H^{2,1}\left(M,\tilde{M}\right)$ and $H^{2,2}\left(M,\tilde{M}\right)$. Theoretical error values were calculated for different parameters of ω and ℓ. The results shown in the tables and graphs are fully consistent with the proven theorems.

CONCLUSIONS

At the current stage of science and technology development, the creation of digital twins is one of the key areas of digital transformation of engineering, industrial and information systems. The functioning of digital twins is inextricably linked to the field of digital signal and image processing, since the construction and updating of such models is based on the analysis of large amounts of data obtained from sensors and measuring systems. This data is often oscillatory in nature, containing high-frequency components and noise, which requires the use of accurate and robust numerical methods for its processing, filtering and integration. In this regard, it is important to expand the range of numerical methods for approximate calculation of integrals of rapidly oscillating functions. Classical integration methods often prove to be computationally expensive or insufficiently accurate in conditions of high oscillation. The development of economical cubature formulas and specialized interpolation schemes allows for a significant reduction in computational costs while maintaining high accuracy, which is critical for the implementation of real-time digital twins.

An important component of modern numerical research is not only the construction of effective cubature formulas, but also a deep understanding of their accuracy properties. In particular, to predict the results of numerical experiments, it is essential to study the dependence of the accuracy of quadrature formulas on the class of differentiability of subintegral functions. This approach allows us to estimate possible errors in advance and to choose integration methods for specific tasks in a reasonable manner.

The scientific novelty of the work lies in the development and theoretical justification of a cubic formula for the numerical integration of two-dimensional rapidly oscillating functions of general form, constructed on the basis of economical interpolation schemes using O.M. Lytvyn's information operators. For the first time, the dependence of the accuracy of this cubature formula on the class of differentiability of subintegral functions has been systematically investigated, and theoretical estimates of approximation errors for different classes of smoothness have been obtained. The established patterns of change in the integration error allow predicting the accuracy of numerical calculations depending on the properties of functions and discretisation parameters, which justifies the feasibility of applying the proposed approach in the tasks of digital signal processing, engineering modeling and digital twin construction.

The practical significance lies in the fact that the established patterns of change in integration error depending on the properties of functions and discretisation parameters allow predicting the accuracy of numerical calculations, which justifies the feasibility of applying the proposed approach in digital image processing, engineering modeling, and digital twin construction tasks.

The prospects for further research should focus on extending the proposed approach to multidimensional fast-oscillating functions and more complex integration domains. Promising areas include research into adaptive cubature formulas, in which the discretisation parameters and the choice of integration nodes are automatically adjusted to take into account the smoothness of the subintegral functions. Such research can be aimed at developing effective algorithms for implementing the proposed methods in digital image processing and digital twin construction tasks.

DECLARATIONS

Conflict of interest: The authors declare that they have no conflict of interest in relation to this research, whether financial, personal, authorship, or otherwise, that could affect the research and its results presented in this paper.

Authors' contributions: Olesia Nehuiviter: formulation of ideas, setting research goals and objectives; scientific supervision of work, mentoring and overall coordination of research; development of cubature formulas; Yevheniia Khurdei: writing the initial draft of the manuscript reviewing, editing and revising the text before submission publication, development algorithms for conducting numerical experiments, verification and confirmation of research results; Vladyslav Ivanov: direct conduct of research and proof of Theorem 2; Ostap Hishchak: direct conduct of research and proof of Theorem 3; Andriy Shnitsar: direct conduct of research and proof of Theorem 4; Anton Zaborniy: preparation of graphs, tables, presentation of results; verification and confirmation of the correctness of the results; Letuta Andrii: preparation of analysis of literary sources, generalisation and systematisation of scientific information.

Data availability: The study does not rely on external datasets; all necessary information is contained in the manuscript.

Software availability: The manuscript has no associated software.

Use of artificial intelligence tools: The authors confirm that they did not use artificial intelligence technologies in creating the submitted work.

REFERENCES

1. Jones D., Snider C., Nassehi A., Yon J., Hicks B. Characterizing the Digital Twin: A systematic literature review. *CIRP Journal of Manufacturing Science and Technology,* 2020, Vol. 29, pp. 36–52. DOI: 10.1016/j.cirpj.2020.02.002
2. Shamshiri S., Liu H., Sohn I. Adversarial robust image processing in medical digital twin. *Information Fusion,* 2025, Vol. 115, 102728. DOI :10.1016/j.inffus.2024.102728
3. Gröhl J., Kunyansky L., Poimala J., Else T., Cecio F., Bohndiek S., Cox B., Hauptmann A. Digital twins enable full-reference quality assessment of photoacoustic image

DOI 10.15588/1607-3274-2026-2-4

reconstructions. *Medical Physics,* 2025. DOI : 10.48550/arXiv.2505.24514
4. Huybrechs D. On the Fourier extension of nonperiodic functions. *SIAM Journal on Numerical Analysis*, 2010, Vol. 47(6), pp. 4326–4355. DOI : 10.1137/090752456
5. D'Ambrosio R., Scalone C. Asymptotic Quadrature Based Numerical Integration of Stochastic Damped Oscillators, *Lecture Notes in Computer Science,* 2021, pp. 622–629. DOI: 10.1007/978-3-030-86960-1_45
6. Gao J., Chang G. A bivariate Filon-Clenshaw-Curtis method of the highly oscillatory integrals on a square. *Journal of Computational and Applied Mathematics*, 2023, 115599. DOI : 10.1016/j.cam.2023.115599
7. Gao J., Iserles A. On an extended Filon method for highly oscillatory integrals over a simplex. *A Mathematics of Computation*, 2023, Vol. 92, pp. 867–893. DOI: 10.1090/mcom/3797
8. Iserles A., Maierhofer G. An accelerated Levin-Clenshaw-Curtis method for the evaluation of highly oscillatory integrals. *BIT Numerical Mathematics*, 2025, Vol. 65, Art. 36. DOI:10.1007/s10543-025-01079-4
9. Schäfer M., Ullrich T. Besov regularity of multivariate non-periodic functions in terms of half-period cosine coefficients and consequences for recovery and numerical integration, *Numerical Analysis,* 2025. DOI : 10.48550/arXiv.2504.03903
10. Iserles A., Nørsett S. P. Efficient quadrature of highly oscillatory integrals using derivatives. *Proceedings of the Royal Society*, 2005, Vol. 461, Iss. 2057, pp. 1383–1399. DOI: 10.1098/rspa.2004.1401
11. Khoromskij B., Veit A. Efficient computation of highly oscillatory integrals by using QTT tensor approximation, *Computational Methods in Applied Mathematics*, 2016, Vol. 16, № 3, pp. 145–159. DOI: 10.1515/cmam-2015-0033
12. Luts L. V. Optimal Calculation of Integrals of Rapidly Oscillating Functions for Some Classes of Differential Functions. *Cybernetics and Systems Analysis*, 2024, Vol. 6, pp. 276–284. DOI: 10.1007/s10559-024-00668-5
13. Lytvyn O. N., Sergienko I. V. New Information Operators in Mathematical Modeling (A Review). *Cybernetics and Systems Analysis*, 2018, Vol. 54, pp. 21–30. DOI: 10.1007/s10559-018-0004-5
14. Sergienko I. V., Zadiraka V. K., Lytvyn O. M. Elements of the General Theory of Optimal Algorithms. Springer, 2021, 378 p. DOI: 10.1007/978-3-030-90908-6
15. Lytvyn O. M., Lytvyn O. G. Analysis of the Results of a Computing Experiment to Restore the Discontinuous Functions of Two Variables Using Projections I. *Cybernetics and Systems Analysis,* 2021, Vol. 57, pp. 754–763. DOI : 10.1007/s10559-021-00400-7
16. Lytvyn O. M., Lytvyn O. G. Analysis of the Results of a Computing Experiment to Restore the Discontinuous Functions of Two Variables Using Projections III. *Cybernetics and Systems Analysis,* 2022, Vol. 58, pp. 372–381. DOI : 10.1007/s10559-022-00469-8
17. Nechuiviter O. P., Iarmosh O. V., Kovalchuk K. H. Numerical calculation of multidimensional integrals depended on input information about the function in mathematical modelling of technical and economic processes. *IOP Conference Series : Materials Science and Engineering*, 2021, 1031 (1), 012059. DOI: 10.1088/1757-899X/1031/1/012059
18. Nechuiviter O. P. Application of the theory of new information operators in conducting research in the field of information technologies. *Information Technologies and Learning Tools*, 2021, № 82 (2), pp. 282–296. DOI: 10.33407/itlt.v82i2.4084
19. Nechuiviter O. P. Cubature formula for approximate calculation integral of highly oscillating function of tree variables (irregular case). *Radio Electronics, Computer Science, Control*, 2020, Vol. 4, pp. 65–73. DOI: 10.15588/1607-3274-2020-4-7
20. Nechuiviter O. P. Approximate Calculation of Double Integrals of Highly Oscillating Functions of General Type Using the Cubature Formula of Optimal Order of Accuracy on a Class of Differentiable Functions. *Cybernetics and Systems Analysis,* 2025, Vol. 61, pp. 589–595. DOI: 10.1007/s10559-025-00794-8
21. Nechuiviter O. P., Ivanov V. V., Shnitsar A. S., Hishchak O. R. Efficient numerical integration of two-dimensional highly oscillating functions of general type. *Bulletin of the National Technical University "KhPI". Series: Mathematical modeling in engineering and technologies*, 2025, № 2(9), pp. 91–103. DOI: 10.20998/2222-0631.2025.02(9).12

Received 14.01.2026.
Accepted 23.04.2026.
Published 26.06.2026.

DOI 10.15588/1607-3274-2026-2-4

УДК 519.6

ВПЛИВ ГЛАДКОСТІ ТА ПАРАМЕТРІВ ДИСКРЕТИЗАЦІЇ НА ТОЧНІСТЬ ЧИСЕЛЬНОГО ІНТЕГРУВАННЯ ДВОВИМІРНИХ ШВИДКООСЦИЛЬОВАНИХ ФУНКЦІЙ

Нечуйвітер О. П. – д-р фіз.-мат. наук, професор, завідувачка кафедри інформаційних комп'ютерних технологій і математики, Навчально-науковий інститут «Українська інженерно-педагогічна академія» Харківського національного університету імені В. Н. Каразіна, Харків, Україна. ROR: https://ror.org/03ftejk10. ORCID: https://orcid.org/0000-0003-2775-8471.

Хурдей Є. Л. – старший викладач кафедри інформаційних комп'ютерних технологій і математики, Навчально-науковий інститут «Українська інженерно-педагогічна академія» Харківського національного університету імені В. Н. Каразіна, Харків, Україна. ROR: https://ror.org/03ftejk10. ORCID: https://orcid.org/0000-0001-8317-8194.

Іванов В. В. – аспірант кафедри інформаційних комп'ютерних технологій і математики, Навчально-науковий інститут «Українська інженерно-педагогічна академія» Харківського національного університету імені В. Н. Каразіна, Харків, Україна. ROR : https://ror.org/03ftejk10. ORCID: https://orcid.org/0009-0003-5379-9370.

Шніцар А. С. – аспірант кафедри інформаційних комп'ютерних технологій і математики, Навчально-науковий інститут «Українська інженерно-педагогічна академія» Харківського національного університету імені В. Н. Каразіна, Харків, Україна. ROR: https://ror.org/03ftejk10. ORCID: https://orcid.org/0009-0008-4834-6971.

Гіщак О. Р. – аспірант кафедри інформаційних комп'ютерних технологій і математики, Навчально-науковий інститут «Українська інженерно-педагогічна академія» Харківського національного університету імені В. Н. Каразіна, Харків, Україна. ROR: https://ror.org/03ftejk10. ORCID: https://orcid.org/0009-0002-9362-3647.

Заборний А. В. – аспірант кафедри інформаційних комп'ютерних технологій і математики, Навчально-науковий інститут «Українська інженерно-педагогічна академія» Харківського національного університету імені В. Н. Каразіна, Харків, Україна. ROR: https://ror.org/03ftejk10. ORCID: https://orcid.org/0009-0006-7026-8215.

Летута А. А. – аспірант кафедри інформаційних комп'ютерних технологій і математики, Навчально-науковий інститут «Українська інженерно-педагогічна академія» Харківського національного університету імені В. Н. Каразіна, Харків, Україна. ROR: https://ror.org/03ftejk10. ORCID: https://orcid.org/0009-0001-4353-4534.

АНОТАЦІЯ

Актуальність. Чисельне інтегрування швидкоосцильованих функцій багатьох змінних є ключовим концептом у задачах інженерного моделювання та цифрової обробки зображень. Незважаючи на наявність різних методів інтегрування, вплив параметрів гладкості та дискретизації на точність наближення залишається недостатньо дослідженим.

Мета роботи – аналіз кубатурної формули, яка застосовує економні схеми інтерполяції, та систематичне вивчення впливу гладкості і параметрів дискретизації на точність чисельного інтегрування.

Метод. Існують методи чисельного інтегрування швидкоосцильованих функцій декількох змінних, які розроблені з використанням інформаційних операторів, що відновлюють проміжні значення функцій за відомими значеннями функції в точках, на лініях, площинах. До таких інформаційних операторів відносимо оператори О. М. Литвина, на основі яких створені економні схеми інтерполяції функцій декількох змінних. Їх застосування при побудові кубатурних формул наближеного обчислення подвійних інтегралів від швидкосцильованих функцій декількох змінних загального виду дозволяє робити обчислення з високою точністю. Основна увага приділена питанню як із зростанням гладкості функції покращуються оцінка похибки чисельного інтегрування двовимірних швидкоосцильованих функцій в загальній формі.

Результати. Досліджена кубатурна формула наближеного обчислення подвійного інтегралу від швидкоосцилюючої функції загального виду.

Висновки. Порівняльний аналіз точності кубатурної формули на різних класах функцій показав, що клас диференційовності функції є визначальним чинником, який впливає на швидкість спадання теоретичної похибки чисельного інтегрування. Економні схеми інтерполяції та вищий рівень гладкості функцій дає суттєве підвищення точності наближеного обчислення інтегралів від двовимірних швидкоосцильованих функцій загального виду.

КЛЮЧОВІ СЛОВА: цифровий двійник, цифрова обробка зображень, математичне моделювання, чисельна інтегрування, кубатурна формула, швидкоосцильовані функції, інтерполяція функцій багатьох змінних, розріджена сітка.

ЛІТЕРАТУРА

1. Characterizing the Digital Twin: A systematic literature review / [D. Jones, C. Snider, A. Nassehi et al.] // CIRP Journal of Manufacturing Science and Technology. – 2020. – Vol. 29. – P. 36–52. DOI: 10.1016/j.cirpj.2020.02.002
2. Shamshiri S. Adversarial robust image processing in medical digital twin / S. Shamshiri, H. Liu, I. Sohn // Information Fusion. – 2025. – Vol. 115, 102728. DOI: 10.1016/j.inffus.2024.102728
3. Digital twins enable full-reference quality assessment of photoacoustic image reconstructions / [J. Gröhl, L. Kunyansky, J. Poimala et al.] // Medical Physics. – 2025. DOI: 10.48550/arXiv.2505.24514
4. Huybrechs D. On the Fourier extension of nonperiodic functions / D. Huybrechs // SIAM Journal on Numerical Analysis. – 2010. – Vol. 47(6). – P. 4326–4355. DOI: 10.1137/090752456
5. D'Ambrosio R. Asymptotic Quadrature Based Numerical Integration of Stochastic Damped Oscillators /

DOI 10.15588/1607-3274-2026-2-4

R. D'Ambrosio, C. Scalone // Lecture Notes in Computer Science. – 2021. – P. 622–629. DOI: 10.1007/978-3-030-86960-1_45
6. Gao J. A bivariate Filon-Clenshaw-Curtis method of the highly oscillatory integrals on a square / J. Gao, G. Chang // Journal of Computational and Applied Mathematics. – 2023. – 115599. DOI : 10.1016/j.cam.2023.115599
7. Gao J. On an extended Filon method for highly oscillatory integrals over a simplex / J. Gao, A. Iserles // A Mathematics of Computation. – 2023. – Vol. 92. – P. 867–893. DOI: 10.1090/mcom/3797
8. Iserles A. An accelerated Levin-Clenshaw-Curtis method for the evaluation of highly oscillatory integrals / A. Iserles, G. Maierhofer // BIT Numerical Mathematics. – 2025. – Vol. 65. – Art. 36. DOI:10.1007/s10543-025-01079-4
9. Schäfer M. Besov regularity of multivariate non-periodic functions in terms of half-period cosine coefficients and consequences for recovery and numerical integration / M. Schäfer, T. Ullrich // Numerical Analysis. – 2025. DOI : 10.48550/arXiv.2504.03903
10. Iserles A. Efficient quadrature of highly oscillatory integrals using derivatives / A. Iserles, S. P. Nørsett // Proceedings of the Royal Society. – 2005. – Vol. 461, Iss. 2057. – P. 1383–1399. DOI: 10.1098/rspa.2004.1401
11. Khoromskij B. Efficient computation of highly oscillatory integrals by using QTT tensor approximation // B. Khoromskij, A. Veit // Computational Methods in Applied Mathematics. – 2016. – Vol. 16, № 3. – P. 145–159. DOI: 10.1515/cmam-2015-0033
12. Luts L. V. Optimal Calculation of Integrals of Rapidly Oscillating Functions for Some Classes of Differential Functions / L. V. Luts // Cybernetics and Systems Analysis. – 2024. – Vol. 6. – P. 276–284. DOI: 10.1007/s10559-024-00668-5
13. Lytvyn O. N. New Information Operators in Mathematical Modeling (A Review) / O. N. Lytvyn, I. V. Sergienko // Cybernetics and Systems Analysis. – 2018. – Vol. 54. – P. 21–30. DOI: 10.1007/s10559-018-0004-5
14. Sergienko I. V. Elements of the General Theory of Optimal Algorithms / I. V. Sergienko, V. K. Zadiraka, O. M. Lytvyn. – Springer, 2021. – 378 p. DOI:10.1007/978-3-030-90908-6
15. Lytvyn O. M. Analysis of the Results of a Computing Experiment to Restore the Discontinuous Functions of Two Variables Using Projections I / O. M. Lytvyn, O. G. Lytvyn // Cybernetics and Systems Analysis. – 2021. – Vol. 57. – P. 754–763. DOI: 10.1007/s10559-021-00400-7
16. Lytvyn O. M. Analysis of the Results of a Computing Experiment to Restore the Discontinuous Functions of Two Variables Using Projections III / O. M. Lytvyn, O. G. Lytvyn // Cybernetics and Systems Analysis. – 2022. – Vol. 58. – P. 372–381. DOI : 10.1007/s10559-022-00469-8
17. Nechuiviter O. P. Numerical calculation of multidimensional integrals depended on input information about the function in mathematical modelling of technical and economic processes / O. P. Nechuiviter, O. V. Iarmosh, K. H. Kovalchuk // IOP Conference Series : Materials Science and Engineering. – 2021. – 1031 (1), 012059. DOI: 10.1088/1757-899X/1031/1/012059
18. Nechuiviter O. P. Application of the theory of new information operators in conducting research in the field of information technologies / O. P. Nechuiviter // Information Technologies and Learning Tools. – 2021. – № 82 (2). – P. 282–296. DOI : 10.33407/itlt.v82i2.4084
19. Nechuiviter O. P. Cubature formula for approximate calculation integral of highly oscillating function of tree variables (irregular case) / O. P. Nechuiviter // Radio Electronics, Computer Science, Control. – 2020. – Vol. 4. – P. 65–73. DOI: 10.15588/1607-3274-2020-4-7
20. Nechuiviter O. P. Approximate Calculation of Double Integrals of Highly Oscillating Functions of General Type Using the Cubature Formula of Optimal Order of Accuracy on a Class of Differentiable Functions / O. P. Nechuiviter // Cybernetics and Systems Analysis. – 2025. – Vol. 61. – P. 589–595. DOI: 10.1007/s10559-025-00794-8
21. Efficient numerical integration of two-dimensional highly oscillating functions of general type / [O. P. Nechuiviter, V. V. Ivanov, A. S. Shnitsar, O. R. Hishchak] // Bulletin of the National Technical University "KhPI". Series: Mathematical modeling in engineering and technologies. – 2025. – № 2(9). – P. 91–103. DOI: 10.20998/2222-0631.2025.02(9).12

DOI 10.15588/1607-3274-2026-2-4